# SOME ABELIAN-BY-NILPOTENT VARIETIES OF p-GROUPS

by

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The results presented in this thesis are my own except where otherwise stated.

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## ABSTRACT


The work reported in this thesis can be conveniently divided into two main sections, the first comprising Chapters 2 and 3, and the second Chapters 4 and 5.

The first section is devoted to metabelian varieties. The principal results of this first section can be summarized in the following way. For any variety $\underline{\underline{V}}$ let $d(\underline{\underline{V}})$ be the minimum value of $k$ such that $\underline{\underline{V}}$ is generated by its free group of rank $k$. In Chapter 2 the value of $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$ is found for all primes $p$ and $\alpha = 1$ and $2$. For $\alpha > 2$ bounds are found within which $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$ must lie and it is conjectured that the lower bound is actually attained.

Chapter 3 looks at the question of distributivity in $\mathrm{lat}(\underline{\underline{A}}_2\underline{\underline{A}}_4)$ and it is shown that the subvariety lattice of $\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_6$ is not distributive.

The second section considers varieties of groups that are abelian-by-nilpotent, and in particular, subvarieties of $\underline{\underline{A}}_p\underline{\underline{T}}_p \wedge \underline{\underline{T}}_p\underline{\underline{A}}_p$, where $\underline{\underline{T}}_p = \underline{\underline{B}}_p \wedge \underline{\underline{N}}_2$ for $p \neq 2$, and $\underline{\underline{T}}_2 = \underline{\underline{B}}_4 \wedge \underline{\underline{N}}_2$. For an odd prime $p$ it is shown that a proper subvariety of $\underline{\underline{A}}_p\underline{\underline{T}}_p \wedge \underline{\underline{T}}_p\underline{\underline{A}}_p$ is either nilpotent or is contained in $[\underline{\underline{A}}_p\underline{\underline{A}}_p, k\,\underline{\underline{E}}]$ for some integer $k$. For $p = 2$ the results are similar, although more complicated. The first step towards these results is to find a basis for $\underline{\underline{T}}_p(F_\infty(\underline{\underline{A}}_p\underline{\underline{T}}_p))$ and this is done in Chapter 4. Using this basis a basis is found for $\underline{\underline{T}}_p(F_\infty(\underline{\underline{A}}_p\underline{\underline{T}}_p \wedge \underline{\underline{T}}_p\underline{\underline{A}}_p))$. The rest of the proof of the above results consists almost entirely of commutator calculations and this is done in Chapter 5.

Throughout this thesis extensive use has been made of commutator calculus, so that it seemed worthwhile to use a special form suitable for use in abelian-by-nilpotent groups. This is done in Chapter 1, which also includes some well-known commutator identities and which is basic to both sections of this thesis.

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CONTENTS

# INTRODUCTION

The work reported in this thesis can be conveniently divided into two sections, the first comprising Chapters 2 and 3, and the second Chapters 4 and 5.

The first section is devoted to metabelian varieties. Possibly the most well-known result in this area is due to D.E. Cohen [11] who has shown that lat($\underline{\underline{A}}\,\underline{\underline{A}}$) has minimum condition. Other authors, such as Warren Brisley ([4] and [5]), R.A. Bryce [10], L.G. Kovács and M.F. Newman [14] and P.M. Weichsel [18] have given descriptions of various sublattices of lat($\underline{\underline{A}}\,\underline{\underline{A}}$).

M.S. Brooks ([6], [7] and [8]) has studied the subvarieties of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$ and has given a complete classification of the non-nilpotent join-irreducible varieties in lat($\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$). He has also shown that lat($\underline{\underline{A}}_3\underline{\underline{A}}_9$) is not distributive. However, as far as the classification of nilpotent join-irreducible varieties in lat($\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$) is concerned, little work has been done, and the problem appears very difficult. This thesis contains a contribution to the theory of the nilpotent subvarieties of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha}$, for $\alpha \geq 1$, although it is not directly related to the classification problem.

The principal results of this first section are summarized in 2.4.16. For any variety $\underline{\underline{V}}$ let $d(\underline{\underline{V}})$ be the minimum value of $k$ such that $\underline{\underline{V}}$ is generated by its free group of rank $k$. For $\alpha = 1$ and 2, we find the value of $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$. For $\alpha > 2$ we find bounds within which $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$ must lie and in 2.5.1 it is conjectured that $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$ actually attains the lower bound that has been found.

Chapter 3 looks at the question of distributivity in lat$(A_2A_4)$. One of the first examples of a non-distributive variety lattice was given by Graham Higman [13] and this raised the question of whether the subvariety lattice of a given variety is distributive or not. R.A. Bryce [10] has shown that lat$(AA)$ is in general not distributive, although many sublattices are distributive. For example, Kovács and Newman [14] have shown that lat$(A_{p^\alpha}A_p)$ is distributive for all primes p and all positive integers $\alpha$. In this chapter an improvement is made on the result of Brooks stated before, by showing that lat$(A_2A_4)$ is not distributive.

Next we turn to varieties of groups that are abelian-by-nilpotent. The main question that has been studied in this area is the question of whether a given variety is hereditarily finitely based, that is, whether all its subvarieties are finitely based. M.R. Vaughan-Lee [18] has shown that for all $n,c \in I$, $AN_c \wedge N_nA$ is hereditarily finitely based, and Bryant and Newman [9] have improved this by showing that for any $c \in I$, $N_{c+1}A \wedge N_2N_c$ is hereditarily finitely based. Brady, Bryce and Cossey [3] have also shown that if $(m,n) = 1$, the subvarieties of $A_m(N_2 \wedge B_n)$ are finitely based. In this section we work in the variety $A_pT_p$ where, for $p \neq 2$, $T_p = N_2 \wedge B_p$, and $T_2 = N_2 \wedge B_4$. It is still not known whether all the subvarieties of $A_pT_p$ are finitely based.

The work of J.M. Brady [2] is perhaps more closely related to the work in this thesis. He showed that the nonmetabelian just-non-Cross varieties which are abelian-by-nilpotent are the varieties $A_pT_q$ where p and q are primes, $(p,q) = 1$. A just-non-Cross variety is a variety that is not generated by a finite group, but all of whose subvarieties are generated by finite groups.

This thesis looks at subvarieties of $\mathbf{A}_p\mathbf{T}_p \wedge \mathbf{T}_p\mathbf{A}_p$. The results obtained are summarized in 5.1.1 and 5.1.2. For $p$ an odd prime we show that a subvariety of $\mathbf{A}_p\mathbf{T}_p \wedge \mathbf{T}_p\mathbf{A}_p$ is either nilpotent or is contained in $[\mathbf{A}_p\mathbf{A}_p, k\,\mathbf{E}]$ for some integer $k$. For $p = 2$ the results are very similar, although slightly more complicated. The first step towards these results is finding a basis for $\mathbf{T}_p(F_\infty(\mathbf{A}_p\mathbf{T}_p))$ and this is done in Chapter 4. Using this we are able to find a basis for $\mathbf{T}_p(F_\infty(\mathbf{A}_p\mathbf{T}_p \wedge \mathbf{T}_p\mathbf{A}_p))$. The rest of the proofs of 5.1.1 and 5.1.2 consists almost entirely of commutator calculations and this comprises the rest of Chapter 5.

Throughout this thesis extensive use has been made of commutator calculus so that it seemed worthwhile to use a special form which is suitable to the situation. The commutator calculus that has been described in Chapter 1 is based on that of Brooks [6], with changes made to accommodate more easily the abelian-by-nilpotent situation. It is also noted that throughout this thesis much inspiration has been gained from the work of Brooks. In Chapter 2 his basis theorem for the derived group of $F_\infty(\mathbf{A}_m\mathbf{A}_n)$ is relied on quite heavily. In Chapter 3 his methods are extended to find the example on non-distributivity in $\mathrm{lat}(\mathbf{A}_2\mathbf{A}_4)$. The philosophy of Chapters 4 and 5 is basically that of Brooks, where his methods and terminology are used, although his results are not used as they deal mainly with metabelian groups.

# NOTATION AND TERMINOLOGY

## General Notation

| | |
|---|---|
| $I$ | the set of non-negative integers |
| $I^+$ | the set of positive integers |
| $\emptyset$ | the empty set |
| $\omega$ | the least infinite ordinal |
| $\Rightarrow$ | logical implication |
| $p$ | an arbitrary prime number |
| $GF(p)$ | the field of integers modulo the prime $p$ |
| $GL(k,p)$ | the group of invertible $k \times k$ matrices with entries in $GF(p)$ |
| $[q]$ | the integer part of the non-negative rational number $q$ |
| supp $\delta$ | Let $S$ be any set. The <u>support</u> of a function $\delta : S \to I$, denoted by supp $\delta$, is defined by $$\text{supp } \delta = \{s \in S \mid \delta(s) \neq 0\}$$ |
| $\binom{n}{r}$ | the binomial coefficient: $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ |

## Groups and Varieties

Notation and terminology generally follows that in Hanna Neumann's book [17]. Note, however, that German letters are here represented by double-underlined Roman letters.

The trivial element of every group is denoted by 1. For the definitions below let $H$ be a group; $H_1, H_2, \ldots$ subgroups of $H$; $h_1, h_2, \ldots$ elements of $H$ with $\underline{\underline{h}} = \{h_1, h_2, \ldots\}$; $r_1, r_2, \ldots \in I$; and $k \in I^+ \setminus \{1\}$.

$H_1 \leq H$ $\quad$ $H_1$ is a subgroup of $H$

$gp(\underline{\underline{h}})$ $\quad$ the subgroup of $H$ generated by $\underline{\underline{h}}$

$\langle\underline{\underline{h}}\rangle$ $\quad$ the fully invariant closure of $\underline{\underline{h}}$ in $H$

$h_1^{h_2}$ $\quad$ $h_2^{-1}\, h_1\, h_2$

$[h_1,h_2]$ $\quad$ $h_1^{-1}\, h_1^{h_2}$

$[h_1,\ldots,h_k]$ $\quad$ defined recursively:

$$[h_1,\ldots,h_k] = [[h_1,\ldots,h_{k-1}],h_k]$$

$[h_1,r_2h_2]$ $\quad$ defined recursively: $[h_1,0h_2] = h_1$,

$$[h_1,r_2h_2] = [[h_1,(r_2-1)h_2],h_2]$$

$[h_1,r_2h_2,\ldots,r_kh_k]$ $\quad$ again defined recursively in the obvious manner

$[H_1,H_2]$ $\quad$ $gp(\{[h_1,h_2] \mid h_1 \in H_1, h_2 \in H_2\})$

$[H_1,r_2H_2]$ $\quad$ defined recursively: similarly to above

$\gamma_c(H)$ $\quad$ defined recursively: $\gamma_1(H) = H$, $\quad \gamma_i(H) = [\gamma_{i-1}(H),H]$

$lat(\underline{\underline{V}})$ $\quad$ the lattice of subvarieties of a variety $\underline{\underline{V}}$

lat $H$ $\quad$ the lattice of verbal subgroups of $H$

If $V$ is a set of words and $\underline{\underline{V}}$ a variety such that $V = \text{laws } \underline{\underline{V}}$, we shall often write $\underline{\underline{V}}(H)$ instead of $V(H)$ to represent the verbal subgroup in $H$ determined by $V$.

The <u>exponent</u> of $H$ is the smallest positive integer $e$ such that $h^e = 1$ for all $h \in H$. If no such integer exists $H$ is said to have infinite exponent.

The <u>exponent</u> of a variety $\underline{\underline{V}}$ is the least positive integer $e$ such that $\underline{\underline{V}} \subseteq \underline{\underline{B}}_e$ or is infinite if no such $e$ exists.

# CHAPTER 1

## Definitions and Notation

This chapter consists mainly of definitions and notation that will be used throughout this thesis. Section 1.1 is devoted mainly to introducing the commutator calculus that will be used in the thesis. The definitions and notation introduced in this section are not completely standard, but they will be convenient for groups in the specific varieties in which we will be working.

In section 1.2 we introduce some well known group identities that will be used throughout the thesis and in section 1.3 we investigate the laws of the varieties in which we will be interested, and some of the identities for groups in these varieties.

### 1.1 Commutator Calculus

The motivation for the approach taken in this thesis comes from M.S. Brooks's thesis [6] where he develops a commutator calculus that is particularly suitable for metabelian groups. Some of Brooks's results will be used in this thesis, and many of his techniques, but the groups in which we shall be working will not always be metabelian, so that some of the definitions and conventions used will be different from those of Brooks so as to be more suitable for this different situation.

Following the convention of Brooks we will distinguish between the formal expression of commutators and the group elements which these formal expressions represent. The word "commutator" will be reserved for the first of the meanings mentioned above, and "commutator-element" will be used for the second. The two will be distinguished notationally by using parentheses in writing commutators and brackets in writing commutator-elements.

The groups to which the commutator calculus will be applied will almost always be metabelian, or abelian-by-nilpotent and the definitions are made with these groups in mind, although most of them are formulated in terms of arbitrary groups.

1.1.1 Definition : Let $H$ be any group. The set $C(H)$ of commutators of $H$ is defined as follows. If $h \in H$, then $h$ is a commutator of weight 1. Assume that commutators of weight $< c$ have been defined for $c > 1$, then commutators of weight $c$ are expressions of the form $(c_i, c_j)$ where $c_i$ and $c_j$ are commutators of weights $r$ and $s$ respectively and $r + s = c$.

1.1.2 Definition : Let $H$ be any group. The value of a commutator $c$, written $[c]$, is defined as follows. If $c$ is a commutator of weight 1, then $c = h$, for some $h \in H$, and we define $[c] = h$. If we have defined $[c]$ for $c$ a commutator of weight $< k$, where $k > 1$, then if $c$ has weight $k$, $c$ can be written $c = (c_i, c_j)$, where $c_i$ and $c_j$ have weights less than $k$, and we define $[c] = [[c_i],[c_j]]$. Any element in $H$ that is the value of some commutator of weight $\geq 2$ in $H$ is called a commutator element.

1.1.3 Definition : Let $H$ be any group. A degree function on $C(H)$ is a function $\delta : C(H) \to I$ whose support

$$\text{supp}\ \delta = \{c \in C(H) \mid \delta(c) \neq 0\}$$

is a finite but non-empty set.

1.1.4 Definition : Let $H$ be any group and let $c_1,\dots,c_k \in C(H)$. Then $(c_1,\dots,c_k)$ is defined recursively by $(c_1,\dots,c_k) = ((c_1,\dots,c_{k-1}),c_k)$. Also, if $r \in I$, $(c\ ,rc_2)$ is defined recursively by $(c_1,0c_2) = c_1$, and $(c_1,rc_2) = ((c_1,(r-1)c_2),c_2)$, and for $r_2,r_3,\dots,r_k \in I$, $(c_1,r_2c_2,\dots,r_kc_k)$ is defined recursively in the obvious manner.

1.1.5 Definition : Let $H$ be any group, and let $\underline{\underline{c}}$ be a totally well-ordered subset of $C(H)$. Let $a,b \in \underline{\underline{c}}$, $a \neq b$, and let $\delta$ be a degree function satisfying $\{a,b\} \subseteq \operatorname{supp}\delta \subseteq \underline{\underline{c}}$. Then we write $(a,b,\delta)$ for the commutator $(a,b,\alpha_1c_1,\dots,\alpha_rc_r)$ in which $\operatorname{supp}\delta = \{c_1,\dots,c_r\}$, $c_1 < c_2 < \dots < c_r$, and

$$\alpha_i = \begin{cases} \delta(c_i) & \text{if } c_i \neq a \text{ or } b \\ \delta(c_i) - 1 & \text{if } c_i = a \text{ or } b \end{cases}$$

Then $P(\underline{\underline{c}})$ is the set of all commutators of the form $(a,b,\delta)$ with $\delta$ satisfying the above conditions. If $p = (a,b,\delta)$ then the value of $p$ will be denoted by $[p] = [[a],[b],\delta]$.

We now define a subset of $P(\underline{\underline{c}})$ denoted by $B(\underline{\underline{c}},n)$. The relevance of this subset will be seen in the result of Brooks quoted immediately after the definition. It will also be important for later work in this thesis.

1.1.6 Definition : Let $H$ be any group and let $\underline{\underline{c}}$ be a totally well-ordered subset of $C(H)$ and let $n \in I^+ \setminus \{1\}$. Then $B(\underline{\underline{c}};n)$ is the set of commutators $(a,b,\delta) \in P(\underline{\underline{c}})$ with the following properties :

(i) $\delta(c) < n$ for all $c \in \underline{\underline{c}} \setminus \{a,b\}$,

(ii) $\delta(a) \leq n$, $\delta(b) \leq n$, and $\delta(a) + \delta(b) < 2n$,

(iii) $b = \min \operatorname{supp} \delta$,

(iv) if $\delta(b) = n$, then $a = \max \operatorname{supp} \delta$.

1.1.7 Theorem (see 3.1 of [8]) : Let $H(n) = F_\infty(\underline{\underline{A}}_m \underline{\underline{A}}_n)$, where $m,n \in I$, $m \neq 1$, $n > 1$, and let $\underline{\underline{h}}(n) = \{h_{ni} : i \in I^+\}$ be a free generating set for $H(n)$ which is well-ordered by its indexing set. Then the derived group $H'(n)$ of $H(n)$ is free abelian of exponent $m$. Further, the valuation mapping $\phi(n) : B(\underline{\underline{h}}(n);n) \to H(n)$ is one-to-one, and $B(\underline{\underline{h}}(n);n)\phi(n)$ is a basis for $H'(n)$.

Another result that will be useful in later sections is the following :

1.1.8 Lemma (see section 1.5 of [6]) : Let $H$ be any group and let $\underline{\underline{c}}$ be a totally ordered subset of $C(H)$ such that $|\underline{\underline{c}}| = r < \infty$. Then $|B(\underline{\underline{c}};n)| = (r-1)(n^r-1)$.

The proof is a straightforward numerical computation and is omitted here.

Before finishing this section we make another definition.

1.1.9 Definition : Let $H$ be any group with a subgroup $A$ that is free abelian of exponent $m$. Let $B$ be a basis for $A$. Then an element $a \in A$ is expressed in normal form when written $a = b_1^{e_1} \ldots b_s^{e_s}$ where $b_1,\ldots,b_s$ are pair-wise distinct members of $B$, and $e_1,\ldots,e_s$ are integers satisfying $e_j \not\equiv 0 \bmod m$ for each $m \in \{1,2,\ldots,s\}$.

## 2.1 Commutator Identities

In this section we introduce some well known identities for commutator elements. They will be used extensively throughout the thesis and often without explicit reference.

The following identities are easily verifiable and are not proved here. They can be found in most standard texts, for example, chapter 10 of Marshall Hall's book [12].

1.2.1 Lemma : Let $G$ be any group, and $x,y,z \in G$. Then

i) $[x,y][y,x] = 1$

ii) $[x,yz] = [x,z][x,y]^z = [x,z][x,y][x,y,z]$

iii) $[xy,z] = [x,z]^y[y,z] = [x,z][x,z,y][y,z]$

iv) $[x,y^{-1},z]^y[y,z^{-1},x]^z[z,x^{-1},y]^x = 1$

v) $[x,y,z][y,z,x][z,x,y]$

$$= [y,x][z,x][z,y]^x[x,y][x,z]^y[y,z]^x[x,z][z,x]^y.$$

1.2.2 Lemma : Let $A,B,C$ be normal subgroups of a group $G$. Then

i) $[AB,C] = [A,C][B,C]$

ii) $[A,BC] = [A,B][A,C]$

iii) $[A,B,C] \leq [B,C,A][C,A,B]$

Proof : (i) and (ii) follow directly from (ii) and (iii) of 1.2.1, and (iii) follows from (iv) of 1.2.1.

1.2.3 Lemma : Let $G$ be any group and $A$ an abelian normal subgroup of $G$.

i) If $a_1,a_2,\ldots \in A$, $g_1,g_2,\ldots \in G$, then

$$[\prod_i a_i,g_1,g_2,\ldots] = \prod_i [a_i,g_1,g_2,\ldots]$$

ii) Let $a \in G$ such that $[a,g] \in A$ for any $g \in G$. For $u,v \in G$

$$[a,u,v] = [a,v,u][u,v,a]^{-1}[a,vu,[u,v]]$$

iii) Let $a \in G$, such that $[a,g] \in A$ for all $g \in G$. Then for $r \in I^+$

$$[a,g^r] = \prod_{i=1}^{r} [a,ig]^{\binom{r}{i}}$$

iv) Let $a \in G$, such that $[a,g] \in A$, for all $g \in G$. Then for $k \in I^+$

$$[a,kg] = \prod_{i=1}^{k} [a,g^i]^{(-1)^{k-i}\binom{k}{i}}$$

v) Let $a \in A, u,v \in G$. Then for $r \in I^+$

$$[a,ruv] = \prod_{i=0}^{r} \prod_{j=r-i}^{r} [a,iu,jv]^{\binom{r}{i}\binom{i}{i+j-r}} \mod [A,G'].$$

<u>Proof</u> : (i) This is a generalization of (iii) of 1.2.1.

(ii) To prove (ii) we use the identity $[a,uv] = [a,vu[u,v]]$ and expand both sides using 1.2.1(ii). The result then follows immediately.

(iii) and (iv) These results follow by induction on $r$ and $k$ respectively.

(v) This identity is proved by induction on $r$. For $r = 1$, it is just 1.2.1(ii). For $r > 1$ we use (ii) to say that $[a,u,v] = [a,v,u] \mod [A,G']$.

## 1.3 <u>Some Abelian-by-Nilpotent Varieties</u>

In this section we investigate the laws of some abelian by nilpotent varieties, and some consequences of their laws.

The following result is well-known but for the sake of completeness, it is proved here.

1.3.1 Lemma : Let $m,n,c \in I$ such that $m \neq 1$, $n > 1$, $c \geq 1$.

Let

$$W(m,n,c) = \{[[x_1,x_2,\ldots,x_{c+1}],[y_1,\ldots,y_{c+1}]],[[x_1,\ldots,x_{c+1}],y^n],$$

$$[x^n,y^n],[x_1,\ldots,x_{c+1}]^m,x^{nm}\}.$$

Then $W(m,n,c)$ is a basis for the laws of $\underline{\underline{A}}_m(\underline{\underline{B}}_n \wedge \underline{\underline{N}}_c)$.

Proof : It is obvious that laws $(\underline{\underline{A}}_m(\underline{\underline{B}}_n \wedge \underline{\underline{N}}_c)) \geq W(m,n,c)$. To prove the reverse inclusion, let $H$ be any group for which $W(m,n,c)(H) = \{1\}$. Then the laws $[[x_1,\ldots,x_{c+1}],[y_1,\ldots,y_{c+1}]]$ and $[x_1,\ldots,x_{c+1}]^m$ ensure that $\underline{\underline{A}}_m(\underline{\underline{N}}_c(H)) = \{1\}$, and the laws $[x^n,y^n]$ and $(x^n)^m$ ensure that $\underline{\underline{A}}_m(\underline{\underline{B}}_n(H)) = \{1\}$. Also, $\underline{\underline{N}}_c(H)$ and $\underline{\underline{B}}_n(H)$ commute elementwise since $[x_1,\ldots,x_{c+1},y^n]$ is a law in $H$. Hence $\underline{\underline{A}}_m((\underline{\underline{B}}_n \wedge \underline{\underline{N}}_c)H) = \underline{\underline{A}}_m(\underline{\underline{B}}_n(H).\underline{\underline{N}}_c(H)) = \{1\}$. Thus for any group $H$, $W(m,n,c)(H) = \{1\}$ implies that $H \in \underline{\underline{A}}_m(\underline{\underline{B}}_n \wedge \underline{\underline{N}}_c)$ and this means that $W(m,n,c)$ is a basis for the laws of $\underline{\underline{A}}_m(\underline{\underline{B}}_n \wedge \underline{\underline{N}}_c)$.

The varieties we will be interested in are those for which $m = p$, $n = p^\alpha$ for $\alpha \in I^+$, and $c = 1$, that is $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha}$, $m = n = p$, $c = 2$ for $p$ an odd prime, that is $\underline{\underline{A}}_p\underline{\underline{T}}_p$, and $\underline{\underline{A}}_2\underline{\underline{T}}_2$ which is the situation when $m = 2$, $n = 4$ and $c = 2$. The following result is immediate from 1.3.1.

1.3.2 Corollary : (i) Let $H \in \underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha}$, then $\underline{\underline{A}}_{p^\alpha}(H)$ is an elementary abelian $p$-group.

(ii) Let $K \in \underline{\underline{A}}_p\underline{\underline{T}}_p$. Then $\underline{\underline{T}}_p(K)$ is an elementary abelian $p$-group.

To prove general results for both of these types of varieties, we shall prove them for the varieties $\underline{\underline{A}}_p(\underline{\underline{B}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$, where $\alpha, c \in I^+$. The results for the specific varieties mentioned above are easily deducible from these results.

1.3.3 Lemma : Let $H \in \underline{\underline{A}}_p(\underline{\underline{B}}_{p\alpha} \wedge \underline{\underline{N}}_c)$ and let $A = (\underline{\underline{B}}_{p\alpha} \wedge \underline{\underline{N}}_c)(H)$ Let $u,v \in H$ such that $[u,h],[v,h] \in A$ for all $h \in H$, and let $w$ be any element of $H$. Then

(i) $[u,p^\beta w] = [u,w^{p^\beta}]$ where $\beta \in I^+$, $1 \leq \beta \leq \alpha$

(ii) $[u,v,p^\alpha w] = 1$

(iii) $[u,v,(p^\alpha-1)u,(p^\alpha-1)v] = 1$

(iv) $[u,p^\alpha w,v] = [v,p^\alpha w,u]$

Proof : (i) By 1.2.3 (iii), $[u,w^{p^\beta}] = \prod_{i=1}^{p^\beta} [u,iv]^{\binom{p^\beta}{i}}$, and for $i \in \{1,\ldots,p^\beta - 1\}$, $\binom{p^\beta}{i} \equiv 0 \bmod p$, and since $A$ has exponent $p$ this gives the required result.

(ii) By 1.3.1 we have

$$1 = [u,v,w^{p^\alpha}]$$

$$= [u,v,p^\alpha w] \text{ by (i) above, giving the required}$$

result.

(iii) By 1.3.1 $[x^{p^\alpha},y^{p^\alpha}]$ is a law in $H$, and hence by (i) we have

$$1 = [u^{p^\alpha},v^{p^\alpha}]$$

$$= [u^{p^\alpha},p^\alpha v]$$

$$= [u,v,(p^\alpha-1)u,(p^\alpha-1)v].$$

(iv) By (ii) above and 1.2.3 (ii) we have

$$
\begin{aligned}
1 &= [u,v,w^{p^\alpha}] \\
&= [u,w^{p^\alpha},v][v,w^{p^\alpha},u]^{-1} \\
&= [u,p^\alpha w,v][v,p^\alpha w,u]^{-1}
\end{aligned}
$$

Here we have used the fact that $[u,w^{p^\alpha}v,[v,w^{p^\alpha}]] = 1$ in $H$. The result follows immediately.

1.3.4 Lemma : Let $H \in \underline{\underline{A}}_p(\underline{\underline{B}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$ and let $A = (\underline{\underline{B}}_{p^\alpha} \wedge \underline{\underline{N}}_c)(H)$. Let $u \in A$ and $v \in H$. Then

$$[u,p^\alpha v] = 1 .$$

Proof : The result follows directly from 1.3.3 since $[u,v^{p^\alpha}] = 1$ in $H$ by 1.3.1.

The next lemma gives some results for groups in $\underline{\underline{A}}_2\underline{\underline{T}}_2$ which are not obvious from the preceding work.

1.3.4 Lemma : Let $H \in \underline{\underline{A}}_2\underline{\underline{T}}_2$ and let $u \in \underline{\underline{T}}_2(H)$, $v,w \in H$. Then

(i) $[u,2v^2] = 1$

(ii) $[u,2[w,v]] = 1$

(iii) $[u,v^2,w] = [u,w,v^2]$.

Proof : (i) By 1.3.1 we have

$$
\begin{aligned}
1 &= [u,v^4] \\
&= [u,2v^2] \quad \text{by 1.2.3 (iii) since } \underline{\underline{T}}_2(H)
\end{aligned}
$$

has exponent 2.

(ii) Again by 1.3.1 we have

$$
\begin{aligned}
1 &= [u,(vw)^4] \\
&= [u,v^4w^4[w,v]^6t] \quad \text{where } t \in \gamma_3(H) \\
&= [u,[w,v]^2] \quad \text{by 1.3.1} \\
&= [u,2[w,v]] \quad \text{by 1.2.3 (iii).}
\end{aligned}
$$

(iii) This result follows directly from 1.2.3 (ii) if we can show that $[v^2,w] \in \underline{\underline{T}}_2(H)$. But $[v^2,w] = [w,w]^2 \bmod \underline{\underline{N}}_2(H)$, and $[v,w]^2 = (wv)^4v^{-4}w^{-4}$ modulo $\underline{\underline{T}}_2(H)$, so that $[v,w]^2 \in \underline{\underline{T}}_2(H)$ and $[v^2,w] \in \underline{\underline{T}}_2(H)$.

# CHAPTER 2

## Generating Groups for $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$

If $\underline{\underline{V}}$ is any variety, let $d(\underline{\underline{V}})$ be the smallest integer $k$ such that the free group of rank $k$ generates $\underline{\underline{V}}$. In this chapter we find a lower bound for $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$ when $\alpha \geq 1$, $c \geq 2$, and for the cases $\alpha = 1,2$ we show that this lower bound is the actual value of $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$. We note that if $\alpha = 0$, or $c = 1$ the varieties concerned are in fact abelian and are generated by their free group of rank one.

In the first section we define a family of words and show that certain of these words are laws in $F_r(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$ for specific values of $r$ and $c$. This allows us to determine a lower bound for $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$. In 2.2 we look at some properties of words if they are to be laws in free groups of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$. One of these properties is "linearity" and in 2.3 we examine the consequences of this condition. In 2.4 using the information of the previous sections, we prove that the lower bound we have found is equal to $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$ for $\alpha = 1,2$.

### 2.1 A Lower Bound for $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$

The main aim of this section is to define a family of words and to show that certain members of this family are laws in some free groups of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$, and therefore that these free groups cannot generate $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$. We treat the case $\alpha = 1$ separately from the case $\alpha \geq 2$, but before we do this there are some preliminary definitions and results.

Throughout this chapter we will rely heavily on Brooks's basis theorem stated in 1.1.7. However, we will use it in a modified form. We will be working mainly in the free groups of $\underline{\underline{A}}_p\underline{\underline{A}}_{p\alpha} \wedge \underline{\underline{N}}_c$, so for convenience of notation throughout the chapter we shall let $G_r(\alpha,c) = F_r(\underline{\underline{A}}_p\underline{\underline{A}}_{p\alpha} \wedge \underline{\underline{N}}_c)$ and we let $\underline{\underline{g}}_r = \{g_1,\ldots,g_r\}$ be a free generating set for $G_r(\alpha,c)$.

We note that if $p = (a,b,\delta) \in P(\underline{\underline{g}}_r)$, then by 1.1.1 weight of $p = \sum_{g\in\underline{\underline{g}}_r} \delta(g)$. We define the set of commutators $B_c(\underline{\underline{g}}_r;p^\alpha)$ as follows :

$$B_c(\underline{\underline{g}}_r;p^\alpha) = \{p \in B(\underline{\underline{g}}_r;p^\alpha) : \text{weight } p \leq c\} .$$

We can now state a modified version of 1.1.7.

2.1.1 <u>Theorem</u> : The derived group $G_r'(\alpha,c)$ of $G_r(\alpha,c)$ is free abelian of exponent $p$. The valuation mapping $\phi_c(p^\alpha,r) : B_c(\underline{\underline{g}}_r;p^\alpha) \to G_r(\alpha,c)$ is one-to-one and $B_c(\underline{\underline{g}}_r;p^\alpha)\phi_c(p^\alpha,r)$ is a basis for $G_r'(\alpha,c)$.

It is readily seen that any counter-example to 2.1.1 would provide a counter-example to 1.1.7.

With this basis theorem we can express elements of $G_r'(\alpha,c)$ in normal form as described in 1.1.9. That is, if $u \in G_r'(\alpha,c)$ we can write $u = b_1^{e_1} \ldots b_s^{e_s}$ where $b_1,\ldots,b_s$ are distinct members of the basis for $G_r'(\alpha,c)$ and $e_1,\ldots,e_s$ are integers satisfying $e_j \not\equiv 0 \pmod p$ for each $j \in \{1,\ldots,s\}$. Such an expression is unique up to the arrangement of the product and congruence modulo $p$ of the indices.

This concept of normal form makes possible the definition of "weight" for elements of $G'_r(\alpha,c)$.

2.1.2 <u>Definition</u> : Let $b$ be a basis element in $G'_r(\alpha,c)$. Then $b = [p]$ for a unique $p \in B_c(\underline{\underline{g}}_r;p^\alpha)$ and we define weight $b$ = weight $p$. Let $u \in G'_r(\alpha,c)$ be expressed in normal form by $u = b_1^{e_1} \ldots b_s^{e_s}$, then we define weight $u = \min(\mathrm{wt}\ b_i \mid i \in \{1,\ldots,s\})$.

We also want to be able to speak of "words" in a free group. Following Hanna Neumann [17] we introduce an alphabet of letters $\underline{\underline{x}} = \{x_1,x_2,\ldots\}$ and we denote by $X$ the free group freely generated by $\underline{\underline{x}}$. This free group will be used to provide "words" : a word is an element of $X$. In this section we will use $w$ to represent a word in the letters (or variables) $x_1,\ldots,x_n$ for some $n \in I^+$. If $A$ is any group, $\theta : \underline{\underline{x}} \to \underline{\underline{x}}\theta \subseteq A$ a mapping of $\underline{\underline{x}}$ into $A$, then the image of the word $w$ under the corresponding homomorphism $\theta : X \to A$ is a value of the word $w$ in $A$. If $x_i\theta = a_i$ for $i \in I^+$ then we shall write $w\theta = w(a_1,\ldots,a_n)$.

We shall sometimes also use the following convention : if $1 \leq k < \ell \leq n$, then we shall write $w(a_1,\ldots,a_n) = w(a_k,a_\ell)$ indicating only the variables in the $k$-th and $\ell$-th places. We now make the following definition :

2.1.3 <u>Definition</u> : Let $w$ be a word in $X$ and let $A$ be any group. Then $w$ is <u>symmetric in $A$</u> if for all $a_k,a_\ell \in A$, $1 \leq k < \ell \leq n$, $w(a_k,a_\ell) = w(a_\ell,a_k)$; $w$ is <u>antisymmetric in $A$</u> if $w(a_k,a_\ell) = w(a_\ell,a_k)^{-1}$. Also $w$ is said to be <u>linear in $A$</u> if for all $a_1,\ldots,a_n$, $g \in A$, and all $i \in \{1,\ldots,n\}$

$$w(a_1,\ldots,a_{i-1},a_ig,a_{i+1},\ldots,a_n) = w(a_1,\ldots,a_n)w(a_1,\ldots,a_{i-1},g,a_{i+1},\ldots,a_n).$$

The following result is a special case of 35.21 of [17].

2.1.4 Lemma : If $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_{c+1}$ is generated by its free group of rank $r$, then $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$ is also generated by its free group of rank $r$.

Please see Corrigenda at back of thesis for proof of this Lemma.

Proof : Suppose $G_r(\alpha,c)$ does not generate $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$. Then there is a word $w$ that is a law in $G_r(\alpha,c)$ but not in $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$, and we may assume that $w$ involves $s$ variables, where $s \leq c$. Form $w^* = [w(x_1,\ldots,x_s),x_{s+1}]$. Then $w^*$ is a law in $G_r(\alpha,c+1)$. But $w^*$ cannot be a law in $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_{c+1}$, for if it were, $w$ would be in the $(c+1)$st term of the lower central series of every free group of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_{c+1}$. In particular $w$ would belong to the $(c+1)$st term of $G_c(\alpha,c)$ and would therefore be a law in $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$ contradicting the choice of $w$.

As an immediate corollary we have :

2.1.5 Corollary : If $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$ is not generated by its free group of rank $r$, then neither is $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_{c+1}$.

We now look at $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$ where $c = p + 2$, $\alpha \geq 2$ and show that this variety is not generated by its free group of rank 2. The results of Brisley, [4] and [5], can be used to show that $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$ is generated by its free group of rank 2 as long as $c \leq p + 1$, and the following result shows that this is the terminal value of $c$.

2.1.6 Lemma : Let

$$w = [x_2,x_1,px_3][x_2,px_1,x_3]^{-1}[x_3,x_1,px_2]^{-1}[x_3,px_1,x_2]$$

Then $w$ is a law in $G_2(\alpha,p+2)$ where $\alpha \in I^+$.

Proof : It is easily seen that if $G \in \underset{=}{A}_p\underset{=}{A}_{p\alpha} \wedge \underset{=}{N}_{p+2}$, then $w$ is antisymmetric and linear in $G$. Thus to show that $w$ is trivial in $G_2(\alpha,p+2)$ it is sufficient to show that $w$ is trivial when $x_1,x_2,x_3$ are replaced by the generators of $G_2(\alpha,p+2)$. But then, two of the variables must be replaced by the same generator, and under this condition it is obvious that $w$ becomes trivial.

We now look at $\underset{=}{A}_p\underset{=}{A}_p \wedge \underset{=}{N}_c$ and show that certain free groups do not generate $\underset{=}{A}_p\underset{=}{A}_p \wedge \underset{=}{N}_c$. The subvariety lattice of $\underset{=}{A}_p\underset{=}{A}_p$ is known (see [14]) and with this information we will have nearly enough to find $d(\underset{=}{A}_p\underset{=}{A}_p \wedge \underset{=}{N}_c)$.

2.1.7 Lemma : Let $w = \prod_{i=2}^{\lambda p} [x_i,x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{\lambda p}]$ where $\lambda \in I^+$. Then $w$ is a law in $G_{kp+1}(1,\lambda p)$ when $\lambda = kp - k + 1$, $k \geq 1$.

Proof : It is easily seen that $w$ is linear and symmetric in $G_{kp+1}(1,\lambda p)$ when $\lambda = kp - k + 1$. So to show that $w$ is a law in $G_{kp+1}(1,\lambda p)$ it is sufficient to show that $w$ is trivial when $x_1,\ldots,x_{\lambda p}$ are replaced by the generators of $G_{kp+1}(1,\lambda p)$. Let $\{g_1,g_2,\ldots,g_{kp+1}\}$ be a free generating set for $G_{kp+1}(1,\lambda p)$. Since $(kp+1)(p-1) = \lambda p - 1$ at least one of the variables must occur $p$ times. Since $w$ is symmetric we may assume without loss of generality that $x_1 = x_2 = \ldots = x_p = g_1$. For $2 \leq i \leq p$, $[x_i,x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{\lambda p}]$ becomes trivial under this condition, and for $i > p$, $[x_i,x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{\lambda p}]$ becomes $[g_{j_i},pg_1,\ldots]$ for some $j_i \in \{1,\ldots,kp+1\}$, so that

$$w(g_1,\ldots,g_1,g_{j_{p+1}},\ldots,g_{j_{\lambda p}}) = \prod_{i=p+1}^{\lambda p} [g_{j_i},pg_1,\ldots]$$

$$= [g_{j_{\lambda p}},pg_1,\ldots]^{\lambda p-p} \quad \text{by 1.3.3(iv)}$$

$$= 1$$

2.1.8 Corollary : Let $k \in I^+$. Then if $c \geq kp^2 - kp + p$, $d(\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_c) \geq kp + 2$.

Proof : The result follows directly from 2.1.7 and 2.1.5, since $w$ is not a law in $\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_c$ for $c = kp^2 - kp + p$, $\lambda = kp - k + 1$.

With this result it will not be very difficult to find $d(\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_c)$ for $c \geq 2$, as the lattice of subvarieties of $\underline{\underline{A}}_p\underline{\underline{A}}_p$ has a very simple structure. However, we shall leave this to a later section.

We now consider $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$ for $\alpha > 1$. In the first chapter we introduced the notation $(a,b,\delta)$ where $\delta$ is a degree function, to represent a commutator in a given group. In this section we define some special degree functions $\delta(\Phi,j,\beta)$ and use these to define words $w(\Phi,j,\beta)$ and $w(j,m,\beta)$.

Let $\Phi = \{i_1,\ldots,i_m\} \subset \{1,2,\ldots,m+j\}$, where $m,j \in I^+$, and $i_r \neq i_s$ for $r \neq s$, $r,s \in \{1,\ldots,m\}$. Let $\text{supp}\ \delta(\Phi,j,\beta) = \{x_1,\ldots,x_{m+j}\}$. For $x_k \in \text{supp}\ \delta(\Phi,j,\beta)$ we define

2.1.9 $$\delta(\Phi,j,\beta)(x_k) = \begin{cases} 1 & \text{if } k \in \Phi \\ p^\beta & \text{otherwise.} \end{cases}$$

We define the words $w(\Phi,j,\beta)$ as follows :

2.1.10 $$w(\Phi,j,\beta) = \prod_{k=1}^{m} [x_{i_k}, x_1, \delta(\Phi,j,\beta)].$$

Note that if $1 \in \Phi$ this product has only $m - 1$ non-trivial terms. The words $w(j,m,\beta)$ are defined in the following way :

2.1.11 $$w(j,m,\beta) = \prod_{\substack{\Phi \subset \{1,\ldots,m+j\} \\ |\Phi| = m}} w(\Phi,j,\beta).$$

We now look at the value of $w(j,m,\beta)$ in certain metabelian groups. We have the following results :

2.1.12 <u>Lemma</u> : Let $G \in \underline{\underline{A}}_p \underline{\underline{A}}_{p^\alpha}$ where $\alpha \geq 2$, and let $\beta$ be an integer, $\beta < \alpha$. Then $w(j,p,\beta)$ is symmetric in $G$.

<u>Proof</u> : We have to show that $w(j,p,\beta)(a_k,a_\ell) = w(j,p,\beta)(a_\ell,a_k)$ for all $a_k,a_\ell \in G$, $1 \leq k < \ell \leq p + j$ and we do this by looking at $w(\Phi,j,\beta)(a_\ell,a_k)$, where $|\Phi| = p$. There are several cases to consider.

i) $k,\ell \in \Phi$.

If $k \neq 1$, it is obvious that $w(\Phi,j,\beta)(a_k,a_\ell) = w(\Phi,j,\beta)(a_\ell,a_k)$. So we consider the case $k = 1 = i_1$, say. By 1.2.3(ii) $[a_{i_s},a_\ell,a_1,\ldots] = [a_{i_s},a_1,a_\ell,\ldots][a_\ell,a_1,a_{i_s},\ldots]^{-1}$ for $\ell \neq i_s \neq 1$, so that $[a_{i_s},a_\ell,\delta(\Phi,j,\beta)] = [a_{i_s},a_1,\delta(\Phi,j,\beta)][a_\ell,a_1,\delta(\Phi,j,\beta)]^{-1}$, and also $[a_1,a_\ell,\delta(\Phi,j,\beta)] = [a_\ell,a_1,\delta(\Phi,j,\beta)]^{-1}$. Thus

$$w(\Phi,j,\beta)(a_\ell,a_k) = \prod_{s=2}^{p}\left\{[a_{i_s},a_1,\delta(\Phi,j,\beta)][a_\ell,a_1,\delta(\Phi,j,\beta)]^{-1}\right\}[a_\ell,a_1,\delta(\Phi,j,\beta)]^{-1}$$

$$= \left\{\prod_{s=2}^{p} [a_{i_s},a_1,\delta(\Phi,j,\beta)]\right\} \times \left\{[a_\ell,a_1,\delta(\Phi,j,\beta)]^{-1}\right\}^p$$

$$= w(\Phi,j,\beta)(a_k,a_\ell).$$

ii) $k,\ell \notin \Phi$.

Again if $k \neq 1$, it is obvious that

$w(\Phi,j,\beta)(a_k,a_\ell) = w(\Phi,j,\beta)(a_\ell,a_k)$. So we consider $k = 1$. By 1.2.3(ii),

$[a_{i_s},p^\beta a_\ell,p^\beta a_1,\ldots] = [a_{i_s},p^\beta a_1,p^\beta a_\ell,\ldots][a_\ell,p^\beta a_1,(p^\beta-1)a_\ell,\ldots]^{-1}$

so that $[a_{i_s},a_\ell,\delta(\Phi,j,\beta)] = [a_{i_s},a_1,\delta(\Phi,j,\beta)][a_\ell,a_1,\delta(\Phi,j,\beta)]^{-1}$. Thus

$$w(\Phi,j,\beta)(a_\ell,a_1) = \prod_{s=1}^{p} \left\{[a_{i_s},a_1,\delta(\Phi,j,\beta)][a_\ell,a_1,\delta(\Phi,j,\beta)]^{-1}\right\}$$

$$= w(\Phi,j,\beta)(a_1,a_\ell)\left\{[a_\ell,a_1,\delta(\Phi,j,\beta)]^{-1}\right\}^p$$

$$= w(\Phi,j,\beta)(a_1,a_\ell).$$

iii) $k \in \Phi$, $\ell \notin \Phi$ or $k \notin \Phi$, $\ell \in \Phi$

If $k \neq 1$, it is obvious that

$$w(\Phi,j,\beta)(a_\ell,a_k) = w(\Phi',j,\beta)(a_k,a_\ell)$$

where $\Phi' = (\Phi \setminus \{k\}) \cup \{\ell\}$, and also

$$w(\Phi',j,\beta)(a_\ell,a_k) = w(\Phi,j,\beta)(a_k,a_\ell).$$

If $k = 1 = i_1$, say, then by 1.2.3(ii)

$[a_{i_s},a_\ell,p^\beta a_1,\ldots] = [a_{i_s},p^\beta a_1,a_\ell,\ldots][a_\ell,p^\beta a_1,a_{i_s},\ldots]^{-1}$ so that if

we put $\Phi' = (\Phi \setminus \{1\}) \cup \{\ell\}$ we have

$$w(\Phi,j,\beta)(a_\ell,a_1) = \prod_{s=2}^{p} \left\{[a_{i_s},a_1,\delta(\Phi',j,\beta)][a_\ell,a_1,\delta(\Phi',j,\beta)]^{-1}\right\}$$

$$= \left\{\prod_{s=2}^{p} [a_{i_s},a_1,\delta(\Phi',j,\beta)]\right\}\left\{[a_\ell,a_1,\delta(\Phi',j,\beta)]^{-1}\right\}^{p-1}$$

$$= w(\Phi',j,\beta)(a_1,a_\ell).$$

In a similar way

$$w(\phi',j,\beta)(a_\ell,a_1) = w(\phi',j,\beta)(a_1,a_\ell),$$

and this completes the proof of the lemma, since the results show that

$$w(j,p,\beta)(a_k,a_\ell) = w(j,p,\beta)(a_\ell,a_k) \quad \text{where} \quad k,\ell \in \{1,2,\ldots,p+j\}$$

2.1.13 Lemma : Let $G \in \underset{=}{A}_p\underset{=}{A}_{p^\alpha} \wedge \underset{=}{N}_{jp^\beta+p}$. Then $w(j,p,\beta)$ is linear in $G$ for $1 \leq \beta \leq \alpha - 1$.

Proof : Since $w(j,p,\beta)$ is symmetric in $G$ by 2.1.12 it will be sufficient to show that it satisfies 2.1.1 for some $i \in \{1,2,\ldots,p+j\}$. To do this we use the following identities. If $u \in \gamma_c(G)$, $a,b,d \in G$, then

1) $[u,pab] = [u,pa][u,pb] \bmod \gamma_{c+p+1}(G)$,

2) $[u,ab] = [u,a][u,b] \bmod \gamma_{c+2}(G)$, and

3) $[ab,d] = [a,d][b,d] \bmod \gamma_3(G)$.

With these identities it is obvious that for any $a_1,\ldots,a_{p+j},b \in G$,

$$w(j,p,\beta)(a_1,\ldots,a_{p+j-1},a_{p+j}b) = w(j,p,\beta)(a_1,\ldots,a_{p+j}) \times w(j,p,\beta)(a_1,\ldots,a_{p+j-1},b).$$

Hence $w(j,p,\beta)$ is linear in $G$.

With these properties we can now prove the following result.

2.1.14 Lemma : For $\alpha \geq 2$, $w(j,p,\alpha-1)$ is a law in $G_r(\alpha,jp^{\alpha-1}+p)$ whenever $r(p-1) + 1 \leq p + j$.

Proof : Let $\{g_1,\ldots,g_r\}$ be a free generating set for $G_r(\alpha,jp^{\alpha-1}+p)$. Since $w(j,p,\alpha-1)$ is linear in $G_r(\alpha,jp^{\alpha-1}+p)$ it is sufficient to show that it becomes trivial when $x_1,\ldots,x_{p+j}$ are replaced by the generators of $G_r(\alpha,jp^{\alpha-1}+p)$. Since $r(p-1)+1 \le p+j$ at least one of the generators must occur $p$ times. Since $w(j,p,\alpha-1)$ is symmetric in $G_r(\alpha,jp^{\alpha-1}+p)$, we may assume without loss of generality that $x_1 = x_2 = \ldots = x_p = g_1$.

We consider the values of $w(\Phi,j,\alpha-1)$ under this condition for all possible $\Phi \subset \{1,\ldots,p+j\}$ where $|\Phi| = p$. Let $\{a_1,\ldots,a_{p+j}\} \subseteq \{g_1,\ldots,g_r\}$, such that $a_1 = \ldots = a_p = g_1$. If $\Phi = \{1,\ldots,p\}$, then trivially $w(\Phi,j,\alpha-1)(a_1,\ldots,a_{p+j}) = 1$.

If $\{1,\ldots,p\} \cap \Phi = \emptyset$, and $\Phi = \{i_1,\ldots,i_p\}$, then

$$\begin{aligned}[a_{i_s},a_1,\delta(\Phi,j,\alpha-1)] &= [a_{i_s},p^\alpha g_1,\ldots] \\ &= [a_{p+j},p^\alpha g_1,\ldots] \quad \text{for } s = 1,\ldots,p \quad \text{by 1.3.3(iv)}\end{aligned}$$

Thus $w(\phi,j,\alpha-1)(a_1,\ldots,a_{p+j}) = [a_{p+j},p^\alpha g_1\ldots]^p$

$$= 1.$$

If $\{1,\ldots,p\} \cap \Phi = \psi$, say, where $|\psi| = k$, $0 < k < p$, then $[a_{i_s},a_1,\delta(\Phi,j,\alpha-1)] = [a_{i_s},((p-k)p+k)g_1,\ldots,]$ for $i_s \in \Phi \setminus \psi$. Now consider $\Phi'$ where $i_s \in \Phi'$, and $\Phi' \cap \{1,\ldots,p\} = \psi'$ such that $|\psi'| = k$ and $\Phi \setminus \psi = \Phi' \setminus \psi'$. Then

$$[a_{i_s},a_1,\delta(\Phi',j,\alpha-1)] = [a_{i_s},((p-k)p+k)g_1,\ldots] = [a_{i_s},a_1,\delta(\Phi,j,\alpha-1)].$$

But there are $\binom{p}{k}$ such sets altogether, and so $[a_{i_s},((p-k)p+k)g_1,\ldots]$ occurs in $w(j,p,\alpha-1)(a_1,\ldots,a_{p+j})$ with exponent $\binom{p}{k} \equiv 0 \bmod p$. This completes the proof of the theorem for this is enough to show that $w(j,p,\alpha-1)$ is a law in $G_r(\alpha,jp^{\alpha-1}+p)$ whenever $r(p-1)+1 \le p+j$.

2.1.15 Corollary : For $c \geq p^\alpha - p^{\alpha-1} + p$, $\alpha \geq 2$,

$$d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c) \geq \left[\frac{c+p^\alpha-p^{\alpha-1}-p}{p^\alpha-p^{\alpha-1}}\right] + 1$$

Proof : If $r \geq 1$, $c = rp^\alpha - p^{\alpha-1} + p$, then $w(r(p-1),p,(\alpha-1))$ is a law in $G_{r+1}(\alpha,c)$ by 2.1.14. Further $w$ is not a law in $G_c(\alpha,c)$ for if we replace $x_i$ by $g_i$ in $w(r(p-1),p,(\alpha-1))$ the resulting commutator element is a product of basis elements in $G_c'(\alpha,c)$ and is therefore non-trivial by 2.1.1. Thus $w(r(p-1),p,(\alpha-1))$ is not a law in $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$ and we conclude that $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$ is not generated by its free group of rank $r + 1$. The result follows by noting that $\dfrac{c+p^\alpha-p^{\alpha-1}-p}{p^\alpha-p^{\alpha-1}} = r + 1$ for $c = rp^\alpha - rp^{\alpha-1} + p$, and applying 2.1.5.

2.1.16 Lemma : Let $\alpha \geq 2$, $p + 2 \leq c \leq p^\alpha - p^{\alpha-1} + p$. Then $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c) \geq 3$.

Proof : By 2.1.6,
$w = [x_2,x_1,px_3][x_2,px_1,x_3]^{-1}[x_3,x_1,px_2]^{-1}[x_3,px_1,x_2]$ is a law in $G_2(\alpha,p+2)$ for $\alpha \in I^+$. If we replace $x_i$ by $g_i$, for $i = 1,2,3$ then the commutator element so obtained is a product of basis elements in $G_3(\alpha,c)$ for $\alpha \geq 2$ and is non-trivial. Thus $w$ is not a law in $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_{p+2}$ for $\alpha \geq 2$, and we conclude that $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_{p+2}) \geq 3$. The rest of the result follows by applying 2.1.5.

## 2.2 Some Properties of Laws in $F_r(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$

To find $k$ such that the free group of rank $k$ generates $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$ it is sufficient to show that there is no word $w$ that is a law in $G_k(\alpha,c)$ but not a law in $G_c(\alpha,c)$. We aim to show, at least in the cases $\alpha = 1,2$, that there can be no such words for certain values of $k$ and $c$. In this section we consider words that have non-trivial values in $G_r(\alpha,c)$ and are laws in $G_{r'}(\alpha,c)$ where $r' < r$, and we find some properties of these words. First we prove a preliminary result.

2.2.1 Lemma : Let $g \in G_r(\alpha,c)$ where $r,\alpha,c \in I^+$, $c \geq 1$. Then $g$ can be written $g = g_1^{\alpha_1} \dots g_r^{\alpha_r} u$, where $0 \leq \alpha_i < p^{\alpha+1}$ and $u \in G_r'(\alpha,c)$.

Proof : The result follows directly from 12.12 of [17] and the fact that $G_r(\alpha,c)$ has exponent $p^{\alpha+1}$.

2.2.2 Lemma : Let $r' < r$ and let $w$ be a law in $G_{r'}(\alpha,c)$ but not in $G_r(\alpha,c)$. Then $w$ takes values in $G_r'(\alpha,c)$.

Proof : Suppose $w\theta \notin G_r'(\alpha,c)$ for some $\theta \in \mathrm{Hom}(X,G_r(\alpha,c))$. Then by 2.2.1 $w\theta$ can be written $w\theta = g_1^{\alpha_1} \dots g_r^{\alpha_r} u$, where $u \in G_r'(\alpha,c)$, $0 \leq \alpha_i < p^{\alpha+1}$ and $\alpha_j > 0$ for some $j \in \{1,\dots,r\}$.

Define a homomorphism $\phi : G_r(\alpha,c) \to G_{r'}(\alpha,c)$ by its action on the generators of $G_r(\alpha,c)$ as follows :

$$g_i\phi = \begin{cases} g_1 & \text{if } i = j \\ 1 & \text{otherwise.} \end{cases}$$

Then $w\theta\phi = g_1^{\alpha_j} \neq 1$, contradicting the assumption that $w$ is a law in $G_{r'}(\alpha,c)$.

Please see Corrigenda for restatement and proof of 2.2.3.

2.2.3 Lemma : Let $r' < r$ and let $w$ be a law in $G_{r'}(\alpha,c)$ but not in $G_r(\alpha,c)$. Then there is a word $w_1$ that is a law in $G_{r'}(\alpha,c)$ such that the values of $w_1$ in $G_r(\alpha,c)$ lie in $\gamma_c(G_r(\alpha,c))$.

<u>Proof</u> : If all the values of $w$ in $G_r(\alpha,c)$ lie in $\gamma_c(G_r(\alpha,c))$ we put $w = w_1$ and we are finished. Otherwise let $k$ be the greatest integer such that the values of $w$ in $G_r(\alpha,c)$ lie in $\gamma_k(G_r(\alpha,c))$. Then $w' = [w,x_{n+1}]$ is also a law in $G_{r'}(\alpha,c)$ and the values of $w'$ in $G_r(\alpha,c)$ are non-trivial and lie in $\gamma_{k+1}(G_r(\alpha,c))$. The result follows by induction.

Since we are considering the values a commutator word $w$ takes in the groups $G_r(\alpha,c)$ we may assume that $w$ is a <u>normal word</u>, that is, we may write $w = b_1^{e_1} \dots b_s^{e_s}$ where $b_i$, $i = 1,\dots,s$ are distinct members of $B(\underline{\underline{x}};p^\alpha)\phi$, where $\phi$ is the valuation mapping $\phi : B(\underline{\underline{x}};p^\alpha) \to X$, and $e_i$ are integers such that $e_i \not\equiv 0 \bmod p$ for $i \in \{1,\dots,s\}$. If $w$ is a commutator word that is not normal, there is a normal word $w'$ such that $w = w' \bmod$ laws $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$ and $w$ and $w'$ have the same values in groups in $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$. When working in the free group $X$, we shall write "$w_1 \equiv w_2$" to mean $w_1 = w_2 \bmod$ laws $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$.

If $w = b_1^{e_1} \dots b_s^{e_s}$ is a normal word, where $b_i = [u_i,v_i,\delta_i]$, then we shall say $w$ is a homogeneous normal word if $\operatorname{supp} \delta_1 = \operatorname{supp} \delta_2 = \dots = \operatorname{supp} \delta_s$.

2.2.4 <u>Lemma</u> : Let $w$ be a homogeneous normal word such that $w$ is a law in $G_r(\alpha,c)$. Then $w$ is a word in at least $r + 1$ variables.

<u>Proof</u> : Suppose $w$ is a word in only $r$ variables $x_1,\dots,x_r$. Then $w(g_1,\dots,g_r)$ is a product of basis elements in $G_r(\alpha,c)$ and is therefore non-trivial, contradicting the assumption that $w$ is a law in $G_r(\alpha,c)$.

The next property we consider will be important in later sections.

2.2.5 Lemma : Let $w$ be a homogeneous normal word that is not a law in $G_r(\alpha,c)$ and whose values in $G_r(\alpha,c)$ are in $\gamma_c(G_r(\alpha,c))$. Then there is a normal word $w' \in \langle w \rangle$ such that $w'$ is linear and non-trivial in $G_r(\alpha,c)$ and whose values in $G_r(\alpha,c)$ are in $\gamma_c(G_r(\alpha,c))$.

Proof : If $w$ is linear in $G_r(\alpha,c)$ we put $w = w'$ and we are finished. Otherwise, if $w$ is a word in $k$ variables, assume $w$ has $t$ non-linear variables, where $1 \leq t \leq k$. We use induction on the number of non-linear variables.

Without loss of generality we may assume that $x_1,\ldots,x_t$ are the non-linear variables. In $w$ replace $x_1$ by $x_1x_{k+1}$ and expand the result using 1.2.3(v). Thus we have

$$w(x_1x_{k+1},x_2,\ldots,x_k) \equiv w(x_1,\ldots,x_k)w(x_{k+1},x_2,\ldots,x_k)w_1(x_1,\ldots,x_{k+1}),$$

where $w_1 \in \langle w \rangle$, $w_1$ has non-trivial values in $G_r(\alpha,c)$ which are in $\gamma_c(G_r(\alpha,c))$ and we may assume $w_1$ is a homogeneous normal word.

For $i > 1$, form $w_i$ from $w_{i-1}$ in the following way : in $w_{i-1}(x_1,\ldots,x_{k+i-1})$ replace $x_1$ by $x_1x_{k+i}$ and expand as before, giving

$$w_{i-1}(x_1x_{k+i},x_2,\ldots,x_{k+i-1}) \equiv w_{i-1}(x_1,\ldots,x_{k+i-1})w_{i-1}(x_{k+i},x_2,\ldots,x_{k+i-1}) \times w_i(x_1,\ldots,x_{k+i}),$$

where $w_i \in \langle w \rangle$, $w_i$ has non-trivial values in $G_r(\alpha,c)$ which are in $\gamma_c(G_r(\alpha,c))$ and we may assume $w_i$ is a homogeneous normal word.

But this process cannot be continued indefinitely, since any $w_i$ can be a word in at most $c$ variables. So eventually we reach an $n \in I^+$ such that

$$w_n(x_1 x_{k+n+1}, x_2, \ldots, x_{k+n}) \equiv w_n(x_1, \ldots, x_{k+n}) w_n(x_{k+n+1}, x_2, \ldots, x_{k+n}),$$

with $k + n \leq c$.

This implies that $w_n$ is linear in $G_r(\alpha, c)$ with respect to $x_1$, and after the following lemma we will show that it is also linear with respect to $x_{k+1}, \ldots, x_{k+n}$. So there are at most $t - 1$ non-linear variables in $w_n$, and induction completes the proof.

2.2.6 Lemma : Let $w_i$, $i = 1, \ldots, n$, be the words described in 2.2.5. Then $w_i$ is symmetric in $G_r(\alpha, c)$ with respect to the variables $x_1, x_{k+1}, \ldots, x_{k+i}$.

Proof : We use induction on $i$ and first note that

$$w(x_1 x_{k+1}, x_2, \ldots, x_k) \equiv w(x_{k+1} x_1, x_2, \ldots, x_k)$$

since the values of $w$ in $G_r(\alpha, c)$ are in $\gamma_c(G_r(\alpha, c))$. From the definition of $w_1$ this immediately gives the result for $i = 1$, namely

$$w_1(x_1, x_{k+1}) \equiv w_1(x_{k+1}, x_1).$$

Assume the result for $j < i$, and consider $w_i(x_1, \ldots, x_{k+i})$. As above we have immediately

$$w_i(x_1, x_{k+i}) \equiv w_i(x_{k+i}, x_1),$$

and, using the induction hypothesis and the definition of $w_i$ it is obvious that $w_i(x_h, x_j) \equiv w_i(x_j, x_h)$ where $k + 1 \leq h < j \leq k + i - 1$. So to prove the lemma it will be sufficient to show that $w_i(x_1, x_{k+i-1}) \equiv w_i(x_{k+i-1}, x_1)$.

Putting $k + i = m$, we have

$$\begin{aligned}
&w_i(x_{m-1},x_1)\\
&\equiv w_i(x_{m-1}x_m,x_2,\ldots,x_{m-2},x_1)w_{i-1}(x_{m-1},x_2,\ldots,x_{m-2},x_1)^{-1}\\
&\quad \times w_{i-1}(x_m,x_2,\ldots,x_{m-2},x_1)^{-1}\\
&\equiv w_{i-2}(x_{m-1}x_mx_1,x_2,\ldots,x_{m-2})w_{i-2}(x_{m-1}x_m,x_2,\ldots,x_{m-2})^{-1}\\
&\quad \times w_{i-2}(x_1,\ldots,x_{m-2})^{-1}w_{i-1}(x_{m-1},x_2,\ldots,x_{m-2},x_1)^{-1}w_{i-1}(x_m,x_2,\ldots,x_{m-2},x_1)^{-1}\\
&\equiv w_{i-2}(x_{m-1},x_2,\ldots,x_{m-2})w_{i-2}(x_mx_1,x_2,\ldots,x_{m-2})w_{i-1}(x_{m-1},x_2,\ldots,x_{m-2},x_mx_1)\\
&\quad \times w_{i-2}(x_{m-1}x_m,x_2,\ldots,x_{m-2})^{-1}w_{i-2}(x_1,\ldots,x_{m-2})^{-1}\\
&\quad \times w_{i-1}(x_{m-1},x_2,\ldots,x_{m-2},x_1)^{-1}w_{i-1}(x_m,x_2,\ldots,x_{m-2},x_1)^{-1}\\
&\equiv w_{i-2}(x_{m-1},x_2,\ldots,x_{m-2})w_{i-2}(x_m,x_2,\ldots,x_{m-2})w_{i-1}(x_m,x_2,\ldots,x_{m-2},x_1)\\
&\quad \times w_{i-1}(x_mx_1,x_2,\ldots,x_{m-1})w_{i-2}(x_{m-1}x_m,x_2,\ldots,x_{m-2})^{-1}\\
&\quad \times w_{i-1}(x_1,\ldots,x_{m-1})^{-1}w_{i-1}(x_1,\ldots,x_{m-2},x_m)^{-1}\\
&\equiv w_{i-1}(x_{m-1},x_2,\ldots,x_{m-2},x_m)^{-1}w_{i-1}(x_mx_1,x_2,\ldots,x_{m-1})\\
&\quad \times w_{i-1}(x_1,\ldots,x_{m-1})^{-1}\\
&\equiv w_i(x_m,x_2,\ldots,x_{m-1},x_1)\\
&\equiv w_i(x_1,\ldots,x_m)\ .
\end{aligned}$$

This completes the proof of the lemma.

The corollary to 2.2.6 is immediate and was used in the proof of 2.2.5.

2.2.7 <u>Corollary</u> : Let $w_n$ be the word described in 2.2.5. Then $w_n$ is linear in $G_r(\alpha,c)$ with respect to the variables $x_1,x_{k+1},\ldots,x_{k+n}$.

Proof : In 2.2.5 we showed that $w_n$ is linear in $G_r(\alpha,c)$ with respect to $x_1$, and the symmetric property of $w_n$ with respect to $x_1,x_{k+1},\ldots,x_{k+n}$ proved in 2.2.6 completes the proof.

2.2.8 Lemma : Let $r' < c \leq r$ and let $w$ be a homogeneous normal word that is a law in $G_{r'}(\alpha,c)$ but not in $G_r(\alpha,c)$. Then there is a homogeneous normal word $w'$ such that $w'$ is a law in $G_{r'}(\alpha,c)$, $w'$ is linear and non trivial in $G_r(\alpha,c)$ and the values of $w'$ in $G_r(\alpha,c)$ are in $\gamma_c(G_r(\alpha,c))$.

Proof : By 2.2.2 and 2.2.3 there is a word $w_1$ that is a law in $G_{r'}(\alpha,c)$ and whose values in $G_r(\alpha,c)$ are in $\gamma_c(G_r(\alpha,c)$, and we may assume that $w_1$ is a homogeneous normal word. We now apply 2.2.5 to find $w' \in < w_1 >$ such that $w'$ is linear and non-trivial in $G_r(\alpha,c)$ and whose values in $G_r(\alpha,c)$ are in $\gamma_c(G_r(\alpha,c)$. But since $w_1$ is a law in $G_{r'}(\alpha,c)$ so is $w'$ and the proof is complete.

## 2.3 Some Words that are Linear in $F_r(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$

We have shown that if there is a word $w$ that is a law in $G_{r'}(\alpha,c)$ but not in $G_r(\alpha,c)$ where $r' < r$, then there is a word $w'$ that is linear and non-trivial in $G_r(\alpha,c)$ but is a law in $G_{r'}(\alpha,c)$. In this section we investigate the form of words that are linear in $G_r(\alpha,c)$. First we make a definition.

2.3.1 Definition : Let $w = b_1^{e_1} \ldots b_t^{e_t}$ be a normal word. Then $w$ is called elementary with degree function $\delta$ if, and only if the words $b_1,\ldots,b_t$ all have the same degree function $\delta$.

Given any normal word $w$, $w$ is equivalent to a product of its elementary parts : $w \equiv w_1 \ldots w_s$, where $w_i$ is elementary with degree function $\delta_i$, $i = 1,\ldots,s$ and $\delta_i \neq \delta_j$ for $i \neq j$.

We now prove the following result :

2.3.2 Lemma : Let $r \geq c$ and let $w$ be a homogeneous normal word that is not a law in $G_r(\alpha,c)$ such that $w(G_r(\alpha,c)) \leq \gamma_c(G_r(\alpha,c))$. Then $w$ is linear in $G_r(\alpha,c)$ if, and only if the elementary parts of $w$ are linear in $G_r(\alpha,c)$.

Proof : Let $w \equiv w_1 \dots w_t$ where $w_i$ is elementary with degree function $\delta_i$, $i = 1,\dots,t$ and $\delta_i \neq \delta_j$ for $i \neq j$. If $w_i$ is linear in $G_r(\alpha,c)$, $i = 1,\dots,t$, then so is $w$.

To prove the reverse conclusion, assume that for some $i \in \{1,\dots,t\}$ $w_i$ is not linear in $G_r(\alpha,c)$. That is if $w$ is a word in $k$ variables there is a $j \in \{1,\dots,k\}$ such that

$$w_i(x_1,\dots,x_jx_{k+1},\dots,x_k) \equiv w_i(x_1,\dots,x_k)w_i(x_1,\dots,x_{k+1},\dots,x_k) \times w_i'(x_1,\dots,x_{k+1})$$

where $w_i'$ has non-trivial values in $G_r(\alpha,c)$ and $w_i'(G_r(\alpha,c)) \leq \gamma_c(G_r(\alpha,c))$. Thus we can write

$$w(x_1,\dots,x_jx_{k+1},\dots,x_k) \equiv \prod_{n=1}^{t} w_n(x_1,\dots,x_jx_{k+1},\dots,x_k)$$

$$\equiv \prod_{n=1}^{t} \{w_n(x_1,\dots,x_k)w_n(x_1,\dots,x_{k+1},\dots,x_k)w_n'(x_1,\dots,x_{k+1})\}$$

$$\equiv w(x_1,\dots,x_k)w(x_1,\dots,x_{k+1},\dots,x_k) \prod_{n=1}^{t} w_n'(x_1,\dots,x_{k+1}),$$

where $w_n'$ may be trivial in $G_r(\alpha,c)$ for $n \neq i$. If, however, $w_n'$ is trivial in $G_r(\alpha,c)$ for all $n \neq i$, then $w$ is not linear in $G_r(\alpha,c)$ and the proof is complete.

So assume $w_1' \dots w_{i-1}'w_{i+1}' \dots w_t'$ is not a law in $G_r(\alpha,c)$ and assume $w_n'$ is a normal word :

$$w'_n = b_{n1}^{e_{n1}} \dots b_{nm_n}^{e_{nm_n}} \quad , \quad n = 1,\dots,t$$

where $b_{ns}$ has degree function $\delta_{ns}$, $1 \leq n \leq t$, $1 \leq s \leq m_n$. Note that $\delta_{ns}(x_h) = \delta_n(x_h)$ for $h \neq j$, $n = 1,\dots,t$.

If $w$ is to be linear in $G_r(\alpha,c)$, $w'_1 \dots w'_t \equiv \prod_{n,s} b_{ns}^{e_{ns}}$ must be a law in $G_r(\alpha,c)$. Therefore, for each $s \in \{1,\dots,m_i\}$ there must be words $b_{uv} = b_{is}$ such that

$$b_{is}^{e_{is}} b_{u_1v_1}^{e_1} \dots b_{u_zv_z}^{e_z} \equiv 1$$

where $1 \leq u_h \leq t$, $1 \leq v_h \leq m_{u_h}$, and $u_h \neq i$ for $h \in \{1,\dots,z\}$. But this implies that $\delta_{is} = \delta_{u_1v_1} = \dots = \delta_{u_zv_z}$, and therefore that

$$\delta_i(x_h) = \delta_{u_n}(x_h) \quad \text{for } h \neq j, \ n = 1,\dots,z.$$

Since $w$ has non-trivial values in $\gamma_c(G_r(\alpha,c))$, $\sum_{n=1}^{k} \delta_i(x_n) = c$, and therefore $\delta_i(x_j) = \delta_{u_n}(x_j)$ for $n = 1,\dots,z$. But then we have that $\delta_i = \delta_{u_n}$ where $u_n \in \{1,\dots,t\}$, $u_n \neq i$, contrary to our original assumption. Therefore we conclude that $w'_1 \dots w'_t$ is not a law in $G_r(\alpha,c)$ if $w'_i$ is not a law in $G_r(\alpha,c)$ and that $w$ is not linear in $G_r(\alpha,c)$.

We now come to the main result of this section.

2.3.3 <u>Theorem</u> : Let $w = b_1^{e_1} \dots b_t^{e_t}$ be a homogeneous normal word in $n$ variables that is elementary with degree function $\delta$, such that $w$ is non-trivial in $G_r(\alpha,c)$ and $w(G_r(\alpha,c) \leq \gamma_c(G_r(\alpha,c))$. If $b_i = [x_{j_i}, x_1, \delta]$, $1 \leq i \leq t$, $j_i \in \{2,\dots,n\}$, then $w$ is linear in $G_r(\alpha,c)$ if, and only if the following conditions hold :

1) $\delta(x_{j_i}) = 1, \ i = 1,\dots,t$

2) for $j \in \{1,\dots,n\}, \ j \neq j_i, \ \delta(x_j) = p^\beta, \ 0 \leq \beta < \alpha$

3) if $\delta(x_1) \neq 1$, then $e_1 + \dots + e_t \equiv 0 \bmod p$.

<u>Proof</u> : We note first that if $\delta(x_1) = p^\alpha$ then $w$ is not linear, for $\delta(x_1) = p^\alpha$ implies $w = [x_n, p^\alpha x_1, \dots]^e$ where $x_n = \max \operatorname{supp} \delta$. If we replace $x_1$ by $x_1 x_{n+1}$ and expand the result it is readily seen that $w$ is not linear in $G_r(\alpha,c)$. If $\delta(x_j) = p^\alpha$ for $1 \neq j \neq j_i$, $i = 1,\dots,t$, then $w$ takes trivial values in $G_r(\alpha,c)$, so we do not consider this case.

1) If $\delta(x_{j_i}) = 1$, $i = 1,\dots,t$, then it is obvious that $w$ is linear in $G_r(\alpha,c)$ with respect to the variables $x_{j_i}$.

Suppose $\delta(x_{j_i}) = k > 1$ for some $i \in 1,\dots,t$. Then

$b_i = [x_{j_i}, ax_1, \dots, (k-1)x_{j_i}, \dots]$ and $b_h = [x_{j_h}, ax_1, \dots, kx_{j_i}, \dots]$.

If we replace $x_{j_i}$ by $x_{j_i} x_{n+1}$ and expand the result to obtain a normal word it is readily seen that $w$ is not linear in $G_r(\alpha,c)$ with respect to $x_{j_i}$ if $k > 1$.

2) For $j \neq j_i$, $i = 1,\dots,t$, if $\delta(x_j) = p^\beta$ $(0 \leq \beta < \alpha)$ it is easily seen that $w$ is linear in $G_r(\alpha,c)$ with respect to $x_j$ and if $\delta(x_j) = k \neq p^\beta$ $(0 \leq \beta < \alpha)$, by replacing $x_j$ by $x_j x_{n+1}$ and expanding the result, it can also be seen that $w$ is not linear in $G_r(\alpha,c)$.

3) If $\delta(x_1) = p^\beta$, $1 \leq \beta < \alpha$, then $b_i = [x_{j_i}, p^\beta x_1, \dots]$. Replace $x_1$ by $x_1 x_{n+1}$ and expand the result to obtain

$$[x_{j_i}, x_1 x_{n+1}, (p^\beta - 1) x_1 x_{n+1} \ldots] \equiv \prod_{s=0}^{p^\beta - 1} [x_{j_i}, x_1, s x_1, (p^\beta - 1 - s) x_{n+1}, \ldots]^{\binom{p^\beta - 1}{s}}$$

$$\times \prod_{s=0}^{p^\beta - 1} [x_{j_i}, x_{n+1}, s x_1, (p^\beta - 1 - s) x_{n+1} \ldots]^{\binom{p^\beta - 1}{s}}$$

$$\equiv [x_{j_i}, p^\beta x_1, \ldots][x_{j_i}, p^\beta x_{n+1}, \ldots]$$

$$\times \prod_{k=0}^{p^\beta - 2} [x_{n+1}, x_1, k x_1, \ldots, (p^\beta - 2 - k) x_{n+1}]^{\binom{p^\beta - 1}{k}}$$

$$\equiv [x_{j_i}, p^\beta x_1, \ldots][x_{j_i}, p^\beta x_{n+1}, \ldots] u(x_1, \ldots, x_{n+1}),$$

$$1 \leq i \leq t.$$

Thus

$$w(x_1\, x_{n+1}, x_2, \ldots, x_n) = \prod_{i=1}^{t} [x_{j_i}, p^\beta x_1 x_{n+1}, \ldots]^{e_i}$$

$$\equiv \prod_{i=1}^{t} [x_{j_i}, p^\beta x_1, \ldots,]^{e_i} [x_{j_i}, p^\beta x_{n+1}, \ldots]^{e_i} u^{e_i}$$

$$\equiv w(x_1, \ldots, x_n) w(x_{n+1}, x_2, \ldots, x_n)\, u^{e_1 + \ldots + e_t}$$

where $u$ is a normal word. Hence, if $\delta(x_1) = p^\beta$, $(1 \leq \beta < \alpha)$, $w$ is linear in $G_r(\alpha, c)$ if, and only if $e_1 + \ldots + e_t \equiv 0 \bmod p$.

2.3.4 Remarks : Before finishing this section it will be convenient to make a few remarks concerning the consequences of this theorem. In section 2.1 we showed that the degree functions $\delta(\Phi, j, \beta)$ give rise to words that are linear in the groups in which we are interested. In the next section we use the observation that the only degree functions that can give rise to normal words that are linear in $G_r(2,c)$ and that have values in $\gamma_c(G_r(2,c))$ are the degree functions $\delta(\Phi,j,1)$ where $|\Phi| + pj = c$. Further, if we assume that $w$ is also homogeneous, and if $\delta(\Phi,j,1)$ and $\delta(\psi,k,1)$ are the degree functions of different elementary parts of $w$, then the identities : $|\Phi| + pj = |\psi| + pk = c$, and $|\Phi| + j = |\psi| + k$, imply that $|\Phi| = |\psi|$ and $k = j$. Words that are linear in $G_r(\alpha,c)$ for $\alpha > 2$ need not be as simple in structure as those for $G_r(2,c)$ and this prevents us from applying the methods in the next section to $G_r(\alpha,c)$ for $\alpha > 2$.

## 2.4 Some Upper Bounds for $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p\alpha} \wedge \underline{\underline{N}}_c)$

By 35.12 of [17] we know that $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p\alpha} \wedge \underline{\underline{N}}_c) \leq c$. In this section we lower this upper bound for $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p\alpha} \wedge \underline{\underline{N}}_c)$, and for $\alpha = 1,2$ we find precise values for $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p\alpha} \wedge \underline{\underline{N}}_c)$.

We first consider $\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_c$. A complete description of the lattice of subvarieties of $\underline{\underline{A}}_p\underline{\underline{A}}_p$ has been given by Kovács and Newman [14], and we describe it as follows. Let $\underline{\underline{N}}_{n*}$ be the subvariety of $\underline{\underline{N}}_n$ defined by the additional law $\prod_{s=2}^{n} [x_s,x_1,\ldots,x_{s-1},x_{s+1},\ldots,x_n]$. Note that $\underline{\underline{N}}_{n*} \supseteq \underline{\underline{N}}_{n-1}$. For each prime $p$ let $I^+(p)$ be the ordered extension of $I^+$ defined by :

$$I^+(p) = \{1,\ldots,p-1,p^*,p,\ldots, pr-1,pr^*,pr,\ldots,\omega\} \quad \text{for } p \text{ odd},$$

$$I^+(2) = \{1,2,3,4^*,4,\ldots, 2r-1,2r^*,2r,\ldots,\omega\}.$$

Then if $\underline{\underline{V}}$ is a subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_p$, for $p$ odd, $\underline{\underline{V}}$ is one of the following varieties : $\underline{\underline{A}}_p$, $\underline{\underline{A}}_{p^2}$, $\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{B}}_p \wedge \underline{\underline{N}}_c$, $c \in \{1,\ldots,p-1,p^*\}$, $\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_c$, $c \in I^+(p)$.

If $\underline{\underline{V}}$ is a subvariety of $\underline{\underline{A}}_2\underline{\underline{A}}_2$, then $\underline{\underline{V}} = \underline{\underline{A}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{N}}_c$ for $c \in I^+(2)$, or $\underline{\underline{V}} = \underline{\underline{A}}_2$ or $\underline{\underline{A}}_4$.

We now look at the variety generated by $F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p)$ for $r > 1$.

2.4.1 <u>Lemma</u> : For $r > 1$, $F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p) \in \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge N_{r(p-1)+1}$, and $F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p) \notin \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge N_{r(p-1)}$.

<u>Proof</u> : By 22.48 of [17] $F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p)$ can be embedded in the wreath product, $F_r(\underline{\underline{A}}_p)$ wr $F_r(\underline{\underline{A}}_p)$, and by 5.1 of [15] this wreath product has class $r(p-1)+1$. Therefore $F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p) \in \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge N_{r(p-1)+1}$. By 1.1.7 $F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p)$ has non-trivial commutator elements of weight $r(p-1)+1$, and therefore $F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p) \notin \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge N_{r(p-1)}$.

2.4.2 <u>Lemma</u> : For $r \not\equiv 1 \bmod p$, $\operatorname{var} F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p) = \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge N_{r(p-1)+1}$.

<u>Proof</u> : By the previous lemma,

$$F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p) \in \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{r(p-1)+1}, \text{ and}$$

$$F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p) \notin \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{r(p-1)}.$$

But when $r \not\equiv 1 \bmod p$, $r(p-1)+1 \not\equiv 0 \bmod p$ and as there is no subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_p$ between $\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{r(p-1)}$ and $\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{r(p-1)+1}$, we conclude that

$$\operatorname{var} F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p) = \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge N_{r(p-1)+1}.$$

2.4.3 <u>Lemma</u> : Let $r \equiv 1 \bmod p$. Then $F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p) / Z(F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p))$ generates $\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{r(p-1)}$.

<u>Proof</u> : By 2.1.7, when $r \equiv 1 \bmod p$, $F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p) \in \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{r(p-1)+1^*}$. But if $H = F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p) / Z(F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p))$, then $H \in \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{r(p-1)}$ and $H \notin \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{r(p-1)-1}$. So we conclude that

$$\operatorname{var} H = \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{r(p-1)}.$$

With these lemmas we can give our final result for $d(\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_c)$.

2.4.4 Lemma : $d(\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_c) = \left[\frac{c-1}{p-1}\right] + 1$, except when $c = r(p-1) + 1$, and $r \not\equiv 1 \bmod p$.

$$d(\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{r(p-1)+1}) = r \quad \text{for } r \not\equiv 1 \bmod p.$$

Proof : The proof follows by applying 2.1.4, 2.1.8 and the previous lemmas.

There is an interesting consequence of these results which we treat in the following results.

2.4.5 Lemma : Let $G$ be a splitting extension of a group $A$ by a group $B$, where $A,B \in \underline{\underline{A}}_p$. If $G \in \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{kp^*}$, then $G \in \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{kp-1}$.

Proof : Every element of $G$ can be written $g = ba$ where $b \in B$, $a \in A$. Since $w = \prod_{i=2}^{kp} [x_i, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{kp}]$ is a law in $G$, then

$$g = \prod_{i=2}^{kp} [b_i a_i, b_1 a_1, \ldots, b_{i-1}a_{i-1}, b_{i+1}a_{i+1}, \ldots, b_{kp}a_{kp}] = 1$$

for all $b_1, \ldots, b_{kp} \in B$, $a_1, \ldots, a_{kp} \in A$. But

$$g = \prod_{i=2}^{kp} [b_i, a_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{kp}][a_i, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{kp}] = 1,$$

and by putting $a_i = 1$, $1 \leq i \leq kp - 1$, we have that $[a, b_1, b_2, \ldots, b_{kp-1}] = 1$ for all $a \in A$, and for all $b_1, \ldots, b_{kp-1} \in B$. But this implies that $[x_1, x_2, \ldots, x_{kp}]$ is a law in $G$, and so $G \in \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{kp-1}$.

2.4.6 <u>Corollary</u> : 1) $\mathrm{var}(C_p \text{ wr } C_p^r) = \underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{r(p-1)+1}$ for $r \geq 1$,

2) $\mathrm{var}\, F_r(\underline{\underline{A}}_p\underline{\underline{A}}_p) = \mathrm{var}(C_p \text{ wr } C_p^r)$ for $r \geq 2$, except when $r \equiv 1 \bmod p$.

<u>Proof</u> : By Liebeck's result [15] $C_p$ wr $C_p^r$ has class $r(p-1) + 1$. Using this the conclusions follow directly from 2.4.4 and 2.4.5.

We now turn again to $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$ where $\alpha \geq 2$, and we find an upper bound for $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$. We again let $G_r(\alpha,c)$ represent $F_r(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$.

2.4.7 <u>Lemma</u> : Let $c,r \in I^+$ such that $c \leq 2r - 1$. Then $\mathrm{var}\, G_{r+1}(\alpha,c) = \mathrm{var}\, G_r(\alpha,c)$.

<u>Proof</u> : We assume that there is a word $w$ that is a law in $G_r(\alpha,c)$ but not in $G_{r+1}(\alpha,c)$ and the result follows by contradiction.

By the comments preceding 2.2.4 and by 33.45 of [17] we may assume that $w$ is a homogeneous normal word, and by 2.2.3 we may assume the values of $w$ in $G_{r+1}(\alpha,c)$ are in $\gamma_c(G_{r+1}(\alpha,c))$. Since $w$ has non-trivial values in $G_{r+1}(\alpha,c)$ we may also assume $w$ is a word in $r + 1$ variables. Let us write $w = b_1^{e_1} \ldots b_s^{e_s}$ where $b_i = [x_{j_i}, x_1, \delta_i]$. Since the values of $w$ in $G_{r+1}(\alpha,c)$ are in $\gamma_c(G_r(\alpha,c))$ and since $c \leq 2r - 1$ we have that $\delta_i(x_j) = 1$ for at least three values of $j$, for each $i \in \{1,\ldots,s\}$.

Let $\underline{\underline{h}} = \{x_j : \delta_1(x_j) = 1\}$. Choose $x_k, x_\ell \in \underline{\underline{h}}$ such that $x_k \neq x_{j_1} \neq x_\ell$, and $k < \ell$. Define a homomorphism $\phi : X \to G_r(\alpha,c)$ as follows :

$$x_i\phi = \begin{cases} x_i & 1 \leq i < \ell \\ x_k & i = \ell \\ x_{i-1} & i > \ell. \end{cases}$$

Then $b_1\phi$ is a basis element in $G_r(\alpha,c)$. For $i \neq 1$, $b_i\phi$ is trivial or is also a basis element in $G_r(\alpha,c)$, but $b_i\phi \neq b_1$ for $i \neq 1$. Thus $w = (b_1\phi)^{e_1} \ldots (b_s\phi)^{e_s} \neq 1$ since $b_1\phi \neq 1$, which contradicts the assumption that $w$ is a law in $G_r(\alpha,c)$.

2.4.8 <u>Corollary</u> : $d(\underset{=}{A}_p\underset{=}{A}_{p\alpha} \wedge \underset{=}{N}_c) \leq \left[\frac{c}{2}\right] + 1$.

<u>Proof</u> : By 2.4.7 if $r = \left[\frac{c}{2}\right] + 1$,

$\operatorname{var} G_r(\alpha,c) = \operatorname{var} G_{r+1}(\alpha,c) = \ldots = \operatorname{var} G_c(\alpha,c) = \underset{=}{A}_p\underset{=}{A}_{p\alpha} \wedge \underset{=}{N}_c$.

For $p = 2$, $\alpha = 2$ this is as far as we may go, and we have the following result.

2.4.9 <u>Lemma</u> : $d(\underset{=}{A}_2\underset{=}{A}_4 \wedge \underset{=}{N}_c) = \left[\frac{c}{2}\right] + 1$.

<u>Proof</u> : By 2.1.15 and 2.4.8 we have

$$\left[\frac{c}{2}\right] + 1 \leq d(\underset{=}{A}_2\underset{=}{A}_4 \wedge \underset{=}{N}_c) \leq \left[\frac{c}{2}\right] + 1.$$

We now consider $\underset{=}{A}_p\underset{=}{A}_{p^2} \wedge \underset{=}{N}_c$ for $p \neq 2$. By 2.1 .15 we know that $\underset{=}{A}_p\underset{=}{A}_{p^2} \wedge \underset{=}{N}_{2p^2-p}$ is not generated by its free group of rank 3 and that $\underset{=}{A}_p\underset{=}{A}_{p^2} \wedge \underset{=}{N}_{2p^2-p-1}$ is not generated by its free group of rank 2. In the next series of lemmas we show that $\underset{=}{A}_p\underset{=}{A}_{p^2} \wedge \underset{=}{N}_{2p^2-p-1}$ is generated by its free group of rank 3, and then by an inductive proof we find $d(\underset{=}{A}_p\underset{=}{A}_{p^2} \wedge \underset{=}{N}_c)$ for $c \in I^+$.

We assume that there is a word $w$ that is a law in $G_3(2,2p^2-p-1)$ but not in $G_r(2,2p^2-p-1)$ where $r \geq 2p^2 - p - 1$, and we arrive at a contradiction. As previously we assume that $w$ is a homogeneous normal word that is linear in $G_r(2,2p^2-p-1)$ and whose values in $G_r(2,2p^2-p-1)$ lie in $\gamma_{2p^2-p-1}(G_r(2,2p^2-p-1))$. Thus we may write $w = b_1^{e_1} \ldots b_s^{e_s}$ where $b_i = [x_{j_i},x_1,\delta_i]$, and by 2.3.4 we may assume that $\delta_i = \delta(\Phi_i,j,1)$ where $|\Phi_i| = p - 1 + np$, $j = 2(p-1) - n$ $(0\leq n\leq 2(p-1))$ and $|\Phi_i| = |\Phi_k|$ for all $i,k \in \{1,2,\ldots,|\Phi_i|+j\}$. In the next two lemmas we consider the case $n = 0$.

2.4.10 <u>Lemma</u> : Let $c = 2p^2 - p - 1$ and let $r \geq c$. Let $w = b_1^{e_1} \ldots b_s^{e_s}$ be a homogeneous normal word that is linear and non-trivial in $G_r(2,c)$ and whose values in $G_r(2,c)$ are in $\gamma_c(G_r(2,c))$. Let $b_i = [x_{j_i},x_1,\delta(\Phi_i,j,1)]$ where $|\Phi_i| = p - 1$, $(1\leq i\leq s)$ and $j = 2(p-1)$. If $w$ is a law in $G_3(2,c)$, then for $i,k \in \{1,\ldots,s\}$ such that $\Phi_i \setminus \{j_i\} = \Phi_k \setminus \{j_k\}$, $e_i \equiv e_k \bmod p$.

<u>Proof</u> : For all $i \in \{1,\ldots,s\}$, write $e_i = e(j_i,\Phi_i)$. This will cause no confusion as $b_i$ is completely determined by $j_i$ and $\Phi_i$. Let $\Phi_i = \{j_i,i_1,\ldots,i_{p-2}\}$ for a fixed $i$, and let $\{1,2,\ldots,|\Phi_i|+j\} \setminus \Phi_i = \{j_k,t_1,\ldots,t_{2p-3}\}$, where $t_1 < t_2 < \ldots < t_{2p-3}$.

Define a homomorphism $\theta : X \to G_3(2,2p^2-p-1)$ by

$$x_{j_i}\theta = g_3$$

$$x_{t_m}\theta = g_3, \quad m = p-1,p,\ldots,2p-3.$$

$$x_{i_m}\theta = \begin{cases} g_1 & \text{if } 1 \in \Phi_i, \quad m = 1,2,\ldots,p-2 \\ g_2 & \text{if } 1 \notin \Phi_i \end{cases}$$

$$x_{t_m}\theta = \begin{cases} g_2 & \text{if } 1 \in \Phi_i \quad m = 1,2,\ldots,p-2 \\ g_1 & \text{if } 1 \notin \Phi_i \end{cases}$$

$x_{j_k}\theta = x_{t_1}, x_i\theta = 1$ otherwise.

Then

$$b_i\theta = \begin{cases} [g_3,(p-2)g_1,(p^2-p)g_2,(p^2-p)g_3] & \text{if } 1 \in \Phi_i \\ [g_3,(p^2-p)g_1,(p-2)g_2,(p^2-p)g_3] & \text{if } 1 \notin \Phi_i . \end{cases}$$

Note that $b_i\theta$ is a non-trivial basis element in $G_3(2,2p^2-p-1)$, and for $j \neq i$, $b_j\theta$ will be trivial in $G_3(2,2p^2-p-1)$ or will be a basis element. Also, $b_u\theta = b_i\theta$ if, and only if $j_u = t_m$ for some $m \in \{p-1,\ldots,2p-3\}$, and $\Phi_u \setminus \{j_u\} = \Phi_i \setminus \{j_i\}$. Thus $w\theta = 1$ implies

$$e(j_i,\Phi_i) + \sum_{\substack{j_u=t_m \\ \Phi_u\setminus\{j_u\}=\Phi_i\setminus\{j_i\}}} e(j_u,\Phi_u) \equiv 0 \bmod p.$$

We now define another homomorphism $\theta' : X \to G_3(2,2p^2-p-1)$ by

$$g_{j_k}\theta' = g_3$$

$$g_{j_i}\theta' = \begin{cases} g_2 & \text{if } 1 \in \Phi_i \\ g_1 & \text{if } 1 \notin \Phi_i \end{cases}$$

$$g_j\theta' = g_j\theta \quad \text{for } j \neq j_k, j_i .$$

If $\Phi_i \setminus \{j_i\} = \Phi_k \setminus \{j_k\}$ a similar consideration gives

$$e(j_k,\theta_k) + \sum_{\substack{j_u=t_m \\ \Phi_u\setminus\{j_u\}=\Phi_i\setminus\{j_i\}}} e(j_u,\Phi_u) \equiv 0 \bmod p.$$

Comparing this with the previous result we have immediately that $e(j_i,\Phi_i) \equiv e(j_k,\Phi_k) \bmod p$ and the proof is complete.

Note that if $\psi \subseteq \{1,\ldots,|\Phi_i|+j\}$ such that $|\psi| = |\Phi_i|$ in the notation of 2.4.10, $u \in \psi$, and $\psi \setminus \{u\} = \Phi_k \setminus \{j_k\}$ for some $k \in \{1,\ldots,s\}$, then $[x_u, x_1, \delta(\psi,j,1)] = b_i$ for some $i \in \{1,\ldots,s\}$, for otherwise from the theorem we would conclude that $e_k \equiv 0 \bmod p$.

2.4.11 Lemma : Let $c = 2p^2 - p - 1$ and let $r \geq c$. Let $w = b_1^{e_1} \ldots b_s^{e_s}$ be a homogeneous normal word that is linear and non-trivial in $G_r(2,c)$ such that $w(G_r(2,c)) \leq \gamma_c(G_r(2,c))$. If $b_i = [x_{j_i}, x_1, \delta(\Phi_i,j,1)]$, where $|\Phi_i| = p - 1$, $j_i \in \Phi_i$, $i = 1,\ldots,s$ and $j = 2(p-1)$, then $w$ is not a law in $G_3(2,2p^2-p-1)$.

Proof : We assume that $w$ is a law, and show that this assumption implies that $e_i \equiv 0 \bmod p$, $i = 1,\ldots,s$.

Let $\Phi_i \setminus \{j_i\} = \psi_i$ $(1 \leq i \leq s)$ and write $e_i = e(\psi_i)$. We can do this since 2.4.10 shows that $\psi_i = \psi_j \Rightarrow e_i = e_j = e(\psi_i)$. For a fixed $i$ let $\{1,2,\ldots,|\Phi_i|+j\} \setminus \Phi_i = \{t_1,\ldots,t_{2p-2}\}$. For each $n \in \{1,\ldots,p-1\}$ we define a homomorphism $\theta_n : X \to G_3(2,c)$ by

$$x_{j_i}\theta_n = g_3$$

$$x_{t_k}\theta_n = \begin{cases} g_1 & \text{if } 1 \notin \psi_i \\ g_2 & \text{if } 1 \in \psi_i \end{cases} \qquad k = 1,\ldots,p-1.$$

$$x_{t_k}\theta_n = \begin{cases} g_2 & \text{if } 1 \notin \psi_i \\ g_1 & \text{if } 1 \in \psi_i \end{cases} \qquad k = p,\ldots,p+n-1$$

$$x_{t_k}\theta_n = g_3 \qquad k = p+n,\ldots,2p-2$$

$$x_k\theta_n = \begin{cases} g_2 & \text{if } 1 \notin \psi_i \\ g_1 & \text{if } 1 \in \psi_i \end{cases} \qquad k \in \psi_i$$

$$x_k\theta_n = 1 \qquad k \geq 3p-2.$$

Then

$$b_i\theta_n = \begin{cases} [g_3(p^2-p)g_1,(pn+p-2)g_2,(p^2-(n+1)p)g_3] & \text{if } 1 \notin \psi_i \\ [g_3,(pn+p-2)g_1,(p^2-p)g_2,(p^2-(n+1)p)g_3] & \text{if } 1 \in \psi_i. \end{cases}$$

Note that $b_i\theta_n$, $n = 1,\ldots,p-1$ is non-trivial and is a basis element in $G_3(2,c)$. Also, for $j \neq i$, $b_j\theta_n$ is trivial or is a basis element in $G_3(2,c)$ and $b_k\theta_n = b_i\theta_n$ if, and only if $b_k = [x_{j_k},x_1,\delta(\Phi_k,j,1)]$ where $x_{j_k} \in \{x_{j_i},x_{t_{p+n}},\ldots,x_{t_{2p-2}}\}$, and $\psi_k \subset \psi_i \cup \{t_p,\ldots,t_{p+n-1}\} = \{k_1,\ldots,k_{p+2-n}\}$ say. Then for each $\psi \subset \{k_1,\ldots,k_{p+2-n}\}$ such that $|\psi| = p-2$, there are $p - n$ possible $x_{j_k}$. Thus $w\theta_n = 1$ implies that

$$(p-n) \sum_{\psi\subset\{k_1,\ldots,k_{p+2-n}\}} e(\psi) \equiv 0 \bmod p.$$

But since $p - n \not\equiv 0 \bmod p$, for $n = 1,\ldots,p-1$, we may write

$$f_{n,0}(k_1,\ldots,k_{p+2-n}) = \sum_{\psi\subset\{k_1,\ldots,k_{p+2-n}\}} e(\psi) \equiv 0 \bmod p,$$

and consider $f_{n,0}(k_1,\ldots,k_{p+2-n})$ as a function of $k_1,\ldots,k_{p+2-n}$. We now form new "functions" as follows : for $0 < m < n$, let

$$f_{n,m}(k_1,\ldots,k_{p+2-n}) = f_{n,m-1}(k_1,\ldots,k_{p+2-n}) - f_{n-1,m-1}(k_1,\ldots,k_{m-1},k_{m+1},\ldots,k_{p+2-n}) .$$

Then obviously, $f_{n,m}(k_1,\ldots,k_{p-2+n}) \equiv 0 \bmod p$.

We now claim that $f_{n,m}(k_1,\ldots,k_{p-2+n})$ is $\sum e(\psi)$ summed over all $\psi$ in $\{k_1,\ldots,k_{p-2+n}\}$ with $k_1,\ldots,k_m \in \psi$. The claim is true by definition for $m = 0$. Assume the result for $m' < m$. Then we have that in $f_{n,m-1}(k_1,\dots,k_{p-2+n})$ there are $\binom{p-2+n-(m-1)}{p-2-(m-1)}$ terms and $\binom{p-2+n-m}{p-2-m+1}$ of these avoid $\psi$ where $k_m \in \psi$. But this is precisely the number of terms in $f_{n-1,m-1}(k_1,\dots,k_{m-1},k_{m+1},\dots,k_{p-2+n})$ and by definition the $\psi$ involved here are only those where $k_m \notin \psi$. Thus all the $\psi$ involved in $f_{n,m}(k_1,\dots,k_{p-2+n})$ contain $k_m$, and by the induction hypothesis and the definition of $f_{n,m}(k_1,\dots,k_{p-2+n})$, $k_1,\dots,k_{m-1} \in \psi$ also, giving the required result.

We now consider $f_{n,m}(k_1,\dots,k_{p-2+n})$ for $n = p - 1$, $m = p - 2$, and we have

$$f_{p-1,p-2}(k_1,\dots,k_{2p-3}) = \sum_{\substack{\psi\subset\{k_1,\dots,k_{2p-3}\} \\ \text{and } k_1,\dots,k_{p-2}\in\psi}} e(\psi)$$

$$= e(\psi) \quad \text{where } \psi = \{k_1,\dots,k_{p-2}\}$$

$$\equiv 0 \bmod p.$$

By varying $\theta_n$, we can make $k_1,\dots,k_{p-2}$ quite arbitrary, and so we have $w\theta_n = 1$, $n = 1,\dots,p-1 \Rightarrow w \equiv 1$.

2.4.12 Lemma : Let $c = 2p^2 - p - 1$ and let $r \geq c$. Let $w = b_1^{e_1} \dots b_s^{e_s}$ be a homogeneous normal word that is linear and non-trivial in $G_r(2,c)$ such that $w(G_r(2,c)) \leq \gamma_c(G_r(2,c))$. If $b_i = [x_{j_i},x_1,\delta(\Phi_i,j,1)]$, where $|\Phi_i| = p - 1 + np$ for some $n \in \{1,\dots,2(p-1)\}$, and $j = 2(p-1) - n$, then $w$ is not a law in $G_3(2,c)$.

Proof : We assume that $w$ is a law in $G_3(2,c)$ and show that this assumption implies $e_i \equiv 0 \bmod p$, $i = 1,\dots,s$.

We consider first the case $1 \leq n < p - 1$. For an arbitrary but fixed $i \in \{1,\ldots,s\}$ let $\{1,2,\ldots,|\Phi_i|+ j\} \setminus \Phi_i = \{t_1,\ldots,t_{2(p-1)-n}\}$, where $t_1 < t_2 < \ldots < t_{2(p-1)-n}$. Define a homomorphism $\theta : X \to G_3(2,c)$ as follows :

$$x_{j_i}\theta = g_3$$

$$x_k\theta = \begin{cases} g_2 & \text{if } 1 \notin \Phi_i \\ g_1 & \text{if } 1 \in \Phi_i \end{cases} \qquad k \in \Phi_i \setminus \{j_i\}$$

$$x_{t_k}\theta = \begin{cases} g_1 & \text{if } 1 \notin \Phi_i \\ g_2 & \text{if } 1 \in \Phi_i \end{cases} \qquad k = 1,\ldots,p-2$$

$$x_{t_k}\theta = g_3 \qquad k = p-1,\ldots,2(p-1)-n,$$

$$x_k\theta = 1 \qquad k > (3+n)(p-1) .$$

Then

$$b_i\theta = \begin{cases} [g_3,(p^2-2p)g_1,(p-2+np)g_2,(p^2-np)g_3] & \text{if } 1 \notin \Phi_i \\ [g_3,(p-2+np)g_1,(p^2-2p)g_2,(p^2-np)g_3] & \text{if } 1 \in \Phi_i . \end{cases}$$

Note that $b_i\theta$ is a non-trivial basis element in $G_3(2,c)$. Also, for $k \neq i$, $b_k\theta$ is either trivial, or is a basis element in $G_3(2,c)$, and $b_k\theta = b_i\theta$ if, and only if $b_k = [x_{j_k},x_1,\delta(\Phi_k,j,1)]$ where $j_k = t_m$ for some $m \in \{p-1,\ldots,2(p-1)-n\}$ and $\Phi_k \setminus \{j_k\} = \Phi_i \setminus \{j_i\}$.

Thus, writing $e_k = e(j_k,\Phi_k)$, $w\theta = 1$ implies

$$e(j_i,\Phi_i) + \sum_{\substack{j_k=t_m \\ \Phi_k\setminus\{j_k\}=\Phi_i\setminus\{j_i\}}} e(j_k,\Phi_k) \equiv 0 \bmod p.$$

We define another homomorphism $\theta' : X \to G_3(2,c)$ by

$$x_{j_i}\theta' = \begin{cases} g_1 & \text{if } 1 \notin \Phi_i \\ g_2 & \text{if } 1 \in \Phi_i \end{cases}$$

$$x_j\theta' = x_j\theta \qquad \text{for } j \neq j_i.$$

Then a similar consideration gives

$$\sum_{\substack{j_k = t_m \\ \Phi_k\setminus\{j_k\}=\Phi_i\setminus\{j_i\}}} e(j_k,\Phi_k) \equiv 0 \bmod p.$$

Comparing these two results we have immediately that $e(j_i,\Phi_i) \equiv 0 \bmod p$ and since $i$ was arbitrary this completes the proof for $n \in \{1,\dots,p-2\}$.

The case for $n = p - 1$ is quite simple. Here $|\Phi_i| = p^2 - 1$, and we define a homomorphism $\theta : X \to G_3(2,c)$ as follows :

let $i$ be arbitrary but fixed, then

$$x_{j_i}\theta = g_3$$

$$x_k\theta = \begin{cases} g_2 & \text{if } 1 \notin \Phi_i \\ g_1 & \text{if } 1 \in \Phi_i \end{cases} \qquad k \in \Phi_i \setminus \{j_i\}$$

$$x_k\theta = \begin{cases} g_1 & \text{if } 1 \notin \Phi_i \\ g_2 & \text{if } 1 \in \Phi_i \end{cases} \qquad \text{otherwise.}$$

Then

$$b_i\theta = \begin{cases} [g_3,(p^2-p)g_1,(p^2-2)g_2] & \text{if } 1 \notin \Phi_i \\ [g_3,(p^2-2)g_1,(p^2-p)g_2] & \text{if } 1 \in \Phi_i. \end{cases}$$

Again $b_i\theta$ is a non-trivial basis element in $G_3(2,c)$, and for $j \neq i$, $b_j\theta$ is either trivial or is a basis element. However, $b_j\theta \neq b_i\theta$ for any $j \neq i$, so that $w = 1 \Rightarrow e_i \equiv 0 \bmod p$, and since $i$ was arbitrary this completes the proof for $n = p - 1$.

The case for $n \in \{p-1,\dots,2p-3\}$ is similar to that for $n \in \{1,\dots,p-2\}$. Here $|\Phi_i| = p^2 - 1 + mp$ where $m \in \{1,2,\dots,p-2\}$, and $j = p - 1 - m$. For an arbitrary but fixed $i$ let $\Phi_i = \{j_i, i_1, \dots, i_{p^2-2+mp}\}$ where $i_1 < i_2 < \dots < i_{p^2-2+mp}$. Define $\theta : X \to G_3(2,c)$ as follows :

$$x_{j_i}\theta = g_3$$

$$x_t\theta = \begin{cases} g_1 & \text{if } 1 \notin \Phi_i \\ g_2 & \text{if } 1 \in \Phi_i \end{cases} \qquad t \in \{1,\dots,|\Phi_i| + j\} \setminus \Phi_i$$

$$x_{i_t}\theta = \begin{cases} g_2 & \text{if } 1 \notin \Phi_i \\ g_1 & \text{if } 1 \in \Phi_i \end{cases} \qquad t = 1,\dots,mp$$

$$x_{i_t}\theta = g_3 \qquad t = mp+1,\dots,p^2-2+mp$$

$$x_t\theta = 1 \qquad t > |\Phi_i| + j\,.$$

Then

$$b_i\theta = \begin{cases} [g_3,(p^2-p-mp)g_1,mpg_2,(p^2-2)g_3] & \text{if } 1 \notin \Phi_i \\ [g_3,mpg_1,(p^2-p-mp)g_2,(p^2-2)g_3] & \text{if } 1 \in \Phi_i \end{cases}$$

and $b_i\theta$ is a non-trivial basis element in $G_3(2,c)$. Also, for $k \neq i$, $b_k\theta$ is either trivial or is a basis element in $G_3(2,c)$ and $b_k\theta = b_i\theta$ if, and only if $b_k = [x_{j_k}, x_1, \delta(\Phi_k,j,1)]$ where $x_{j_k} = x_{i_t}$ for some $t \in \{mp+1,\dots,p^2-2+mp\}$ and $\Phi_k = \Phi_i$. Thus, writing $e_i = e(j_i,\Phi_i)$, $w\theta = 1$ implies

$$e(j_i,\Phi_i) + \sum_{t=mp+1}^{p^2-2+mp} e(i_t,\Phi_i) \equiv 0 \bmod p.$$

We define another homomorphism $\theta' : X \to G_3(2,c)$ as follows :

$$x_{j_i}\theta' = \begin{cases} g_2 & \text{if } 1 \notin \Phi_i \\ g_1 & \text{if } 1 \in \Phi_i \end{cases}$$

$$x_j\theta' = x_j\theta \quad \text{for } j \neq j_i.$$

A similar consideration then gives

$$\sum_{t=mp+1}^{p^2-2+mp} e(i_t,\Phi_i) \equiv 0 \bmod p.$$

Comparing these sums we have immediately that $e_i \equiv 0 \bmod p$, and since $i$ was arbitrary this completes the proof for $n \in \{p-1,\ldots,2p-3\}$.

The only case we have not considered is when $|\Phi_i| = 2p^2 - p - 1$, but under these circumstances it can easily be shown that $w$ is not a law in $G_3(2,c)$.

We can now prove our main result here.

2.4.13 <u>Theorem</u> : $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2} \wedge \underline{\underline{N}}_{2p^2-p-1}) = 3.$

<u>Proof</u> : The previous lemmas show that $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2} \wedge \underline{\underline{N}}_{2p^2-p-1}) \leq 3$, for otherwise there would be a word $w$ that is a law in $G_3(2,2p^2-p-1)$ but is not a law in $G_r(2,2p^2-p-1)$ for $r \geq 2p^2 - p - 1$. However, from the results of section 2.2 we may assume that $w$ is a homogeneous normal word that is linear in $G_r(2,2p^2-p-1)$ such that $w(G_r(2,2p^2-p-1)) \leq \gamma_{2p^2-p-1}(G_r(2,2p^2-p-1))$. But by 2.4.11 and 2.4.12 such a word $w$ cannot be a law in $G_3(2,2p^2-p-1)$ and the conclusion follows. By 2.1.15 we also have that $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2} \wedge \underline{\underline{N}}_{2p^2-p-1}) \geq 3$ and this completes the proof of the theorem.

With this theorem we can extend our results to include $c > 2p^2 - p - 1$.

2.4.14 <u>Theorem</u> : For $r \geq 2$, $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2} \wedge \underline{\underline{N}}_{r(p^2-p)+p-1}) = r + 1$.

<u>Proof</u> : By 2.1.15 we have already that $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2} \wedge \underline{\underline{N}}_{r(p^2-p)+p-1}) > r$ and so we have only to show the reverse inequality. The proof is by induction on $r$. For $r = 2$ the result is just 2.4.13, so we assume the result for $r' < r$.

Let $c = r(p^2-p) + p - 1$. If $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2} \wedge \underline{\underline{N}}_c$ is not generated by its free group of rank $r + 1$ then there is a homogeneous normal word $w$ that is linear and non-trivial in $G_c(2,c)$ and whose values in $G_c(2,c)$ are in $\gamma_c(G_c(2,c))$ and is a law in $G_{r+1}(2,c)$. Let $w = b_1^{e_1} \ldots b_t^{e_t}$. Then $b_1$ is equivalent to one of the following forms :

1) $b_1 \equiv [b_1', px_{k_1}, \ldots, px_{k_{p-1}}]$ $\quad 1 < k_1 < \ldots < k_{p-1}$

or 2) $b_1 \equiv [b_1', x_{i_1}, \ldots, x_{i_{p^2-p}}]$, $\quad 1 < i_1 < \ldots < i_{p^2-p}$

where in each case $b_1'$ is the value of a commutator in $B(\underline{\underline{x}};p^2)$.

We consider the first possibility. Suppose $b_1, \ldots, b_s$ are such that $b_i \equiv [b_i', px_{k_1}, \ldots, px_{k_{p-1}}]$, $1 \leq i \leq s$, and that for $j > s$, $b_j$ is not equivalent to a word of this form. Then

$$w \equiv \prod_{i=1}^{s} [b_i', px_{k_1}, \ldots, px_{k_{p-1}}]^{e_i} \times \prod_{i=s+1}^{t} b_i^{e_i}$$

$$\equiv [w', px_{k_1}, \ldots, px_{k_{p-1}}] \times \prod_{i=s+1}^{t} b_i^{e_i},$$

where $w' \equiv b_s'^{e_s} \ldots b_s'^{e_s} \not\equiv 1$. By our induction hypothesis there is a homomorphism $\theta : X \to G_r(2,c)$ such that $w'\theta \neq 1$. We define another homomorphism $\theta' : X \to G_{r+1}(2,c)$ by

$$x_{k_i}\theta' = g_{r+1} \qquad i = 1,\dots,p-1$$

$$x_i\theta' = x_i\theta \qquad \text{for } i \notin \{k_1,\dots,k_{p-1}\}.$$

Then $w\theta' = [w'\theta,(p^2-p)g_{r+1}] \times \prod_{i=s+1}^{t} (b_i\theta')^{e_i}$, and $[w\theta',(p^2-p)g_{r+1}] \neq 1$ and can be written as a product of basis elements in $G_{r+1}(2,c)$ each of which has $g_{r+1}$ occurring with multiplicity $p^2 - p$. Also, $\prod_{i=s+1}^{t} (b_i\theta')^{e_i}$ can be written as a product of basis elements in $G_{r+1}(2,c)$ but in each of these basis elements $g_{r+1}$ occurs strictly less than $p^2 - p$ times. Thus $\prod_{i=s+1}^{t} (b_i\theta')^{e_i} \neq [w'\theta,(p^2-p)g_{r+1}]^{-1}$ and so we conclude $w\theta' \neq 1$ contradicting the assumption that $w$ is a law in $G_{r+1}(2,c)$.

If the second possibility for $b_1$ occurs a similar argument shows that there is a homomorphism $\theta : X \to G_{r+1}(2,c)$ such that $w\theta \neq 1$. Thus $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2} \wedge \underline{\underline{N}}_c$ is generated by its free group of rank $r + 1$, or $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2} \wedge \underline{\underline{N}}_{r(p^2-p)+p-1}) \leq r + 1$, and the proof of the theorem is complete.

The following theorem could be deduced from the results of Brisley, [4] and [5], but for the sake of completeness we prove it here. The proof given is independent of Brisley's results.

2.4.15 Theorem : $\underline{\underline{A}}_p\underline{\underline{A}}_{p^{\alpha+1}} \wedge \underline{\underline{N}}_{p+1}$ is generated by its free group of rank 2.

Proof : We assume the theorem is false and arrive at a contradiction. If $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_{p+1}$ is not generated by its free group of rank 2 there is a word $w$ that is a law in $G_2(\alpha,p\underline{+}1)$ but not in $G_{p+1}(\alpha,p+1)$. The results of section 2.2 allow us to assume that $w$ is a homogeneous normal word that is linear in $G_{p+1}(\alpha,p+1)$ and that the values of $w$ in $G_{p+1}(\alpha,p+1)$ are in $\gamma_{p+1}(G_{p+1}(\alpha,p+1))$. Let $w$ be written as a normal word, $w = b_1^{e_1} \ldots b_t^{e_t}$. But if $w$ is to be linear in $G_{p+1}(\alpha,p+1)$ then by 2.3.3 $b_i$ must be of the form

$$b_i = [x_{j_i}, x_1, \ldots, x_{j_i-1}, x_{j_i+1}, \ldots, x_{p+1}]$$

For $i = 1,\ldots,s$ define homomorphisms $\theta_i : X \to G_2(\alpha,p+1)$ by

$$x_{j_i}\theta_i = g_2$$

$$x_j\theta_i = g_1 \qquad \text{for } j \neq j_i .$$

Then $w\theta_i = [g_2, pg_1]^{e_i} \neq 1$, contradicting the assumption that $w$ is a law in $G_2(\alpha,p+1)$.

2.4.16 <u>Summary</u> : With the help of 2.1.4 the results of this chapter can be summarized as follows :

1) $d(\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_c) = \left[\frac{c-1}{p-1}\right] + 1$, except when $c = r(p-1) + 1$ and $r \not\equiv 1 \bmod p$. In this case $d(\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_{r(p-1)+1}) = r$.

2) For $\alpha \geq 1$, $c \leq p + 1$, $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c) = 2$.

3) For $c \in \{p+2,\ldots,p^2-1\}$, $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2} \wedge \underline{\underline{N}}_c) = 3$.

4) For $c \geq p^2$, $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2} \wedge \underline{\underline{N}}_c) = \left[\frac{c+p^2-2p}{p^2-p}\right] + 1$

5) For $\alpha > 2$, $c \in \{p+2,\ldots,p^\alpha-p^{\alpha-1}+p-1\}$, $3 \leq d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c) \leq \left[\frac{c}{2}\right] + 1$.

6) For $\alpha > 2$, $c \geq p^{\alpha} - p^{\alpha-1} + p$,

$$\left[\frac{c+p^{\alpha}-p^{\alpha-1}-p}{p^{\alpha}-p^{\alpha-1}}\right] + 1 \leq d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c) \leq \left[\frac{c}{2}\right] + 1.$$

## 2.5 Further Remarks

I am confident that the upper bound for $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$ can be reduced considerably, and in fact to the value of the lower bound and I express this conjecture formally as follows :

2.5.1 Conjecture : Let $\alpha \geq 2$, $c \geq p^{\alpha} - p^{\alpha-1} + p$. Then

$$d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c) = \left[\frac{c+p^{\alpha}-p^{\alpha-1}-p}{p^{\alpha}-p^{\alpha-1}}\right] + 1 .$$

The conjecture is true for $\alpha = 2$ as we have already shown. The upper bound for $d(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$ as stated in 2.4.16(6) was proved in 2.4.7 and 2.4.8. I am sure that 2.4.7 could be improved greatly without too much trouble, but to get the result stated in 2.5.1 might be more difficult.

It would also seem that the laws introduced in section 2.1 are quite significant and may give some lead to an investigation of the nilpotent join-irreducible varieties in the lattice of subvarieties of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$. For instance, it can be shown that $\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_4$ is not join irreducible, but is a join of var $F_2(\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_4)$ which has the law $w(1,2,1) = [x_2,2x_1,x_3][x_2,x_1,2x_3][x_3,2x_1,x_2][x_3,x_1,2x_2]$, and the variety in $\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_4$ with the additional law $[x_2,x_1,2x_3][x_3,x_1,2x_2]$. Both of these varieties are maximal in $\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_4$, but var $F_2(\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_4)$ is not join irreducible. I am not sure to what extent these statements can be generalized, but it certainly seems that some generalization is likely.

# CHAPTER 3

## A Non-Distributive Variety Lattice

In this chapter we present an example to show that the lattice of subvarieties of $\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_6$ is not distributive. This answers a question left open by Brooks ([6] and [7]). Brooks showed that lat $(\underline{\underline{A}}_3\underline{\underline{A}}_9 \wedge \underline{\underline{N}}_{11})$ is not distributive and conjectured that lat $(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2} \wedge \underline{\underline{N}}_{p^2})$ is not distributive for all primes $p \geq 3$. For $p = 2$, however, his constructions failed so that the question of the distributivity of lat $(\underline{\underline{A}}_2\underline{\underline{A}}_4)$ was left open.

Here the method used is a generalization of that of Brooks, but instead of considering the verbal subgroup lattice of the free group of rank two, we use the free group of rank three. The example we provide occurs among the subgroups of the last term of the lower central series of $F_3(\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_6)$ and in section 3.1 we characterize the fully invariant subgroups of the last term of the lower central series of such a group. In section 3.2 we provide the example of non-distributivity.

### 3.1 Some Fully Invariant Subgroups

Throughout this chapter we use the notation lat $\underline{\underline{V}}$ and lat $G$ to denote respectively the lattice of subvarieties of a variety $\underline{\underline{V}}$ and the lattice of verbal subgroups of a group $G$.

The following theorem of Brooks [7] gives the relationship between non-distributivity in the lattice of verbal subgroups of a relatively free group $G$, and non-distributivity in lat (var $G$). We will not prove this theorem here.

3.1.1 Theorem : Let $G$ be a relatively free group. If lat $G$ is not distributive, then neither is lat (var $G$). In fact, if for some $C, D_1, D_2 \in \text{lat } G$,

$$C \cap D_1D_2 \neq (C\cap D_1)(C\cap D_2),$$

then

$$\underline{\underline{U}} \vee (\underline{\underline{M}}_1 \wedge \underline{\underline{M}}_2) \neq (\underline{\underline{U}} \vee \underline{\underline{M}}_1) \wedge (\underline{\underline{U}} \vee \underline{\underline{M}}_2) \ ,$$

where $\underline{\underline{M}}_i = \text{var } (G/D_i)$, $i = 1,2$, and $\underline{\underline{U}}$ is any variety for which $\underline{\underline{U}}(G) = C$.

So to demonstrate non-distributivity in $\text{lat } (\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_6)$ it is sufficient to find verbal subgroups of some free group of $\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_6$ which satisfy the conditions of 3.1.1. Our first step towards this is a characterization of some verbal subgroups of relatively free groups. We begin with the following result, but first we make a definition.

3.1.2 Definition : Let $G$ be a relatively free group with free generating set $\{g_1,\ldots,g_r\}$. Then for $k \in \{1,\ldots,r\}$, the deletions $\chi_k$ of $G$ are defined by

$$g_k\chi_k = 1, \text{ and } g_i\chi_k = g_i \quad \text{for } i \neq k.$$

3.1.3 Theorem : Let $G_k(\alpha,c) = F_k(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c)$ where $\alpha, c, k \in I^+$, $\alpha \geq 1$, $c \geq 2$, $k \geq 2$. Then the fully invariant subgroups of $G_k(\alpha,c)$ that are contained in $\gamma_c(G_k(\alpha,c))$ are precisely those subgroups that are closed under the automorphisms and deletions of $G_k(\alpha,c)$.

Proof : Let $M = \underline{\underline{A}}_p(G_k(\alpha,c))$ and let $\eta$ be any endomorphism of $G_k(\alpha,c)$. Denote by $\eta/M$ the endomorphism of $G_k(\alpha,c)/M$ induced by $\eta$. We note that $G_k(\alpha,c)/M$ may be regarded as a $k$-dimensional vector space over $GF(p)$, and that $\text{Aut } (G_k(\alpha,c)/M) \cong GL(k,p)$. Let

$S$ be a subgroup of $\gamma_c(G_k(\alpha,c))$ that is closed under the automorphisms and deletions of $G_k(\alpha,c)$.

If $\ker \eta/M = \{1\}$, then $\eta/M \in \mathrm{Aut}\,(G_k(\alpha,c))$ and $\eta$ is in fact an automorphism of $G_k(\alpha,c)$ since $M$ is the Frattini subgroup of $G_k(\alpha,c)$. Hence $S$ admits $\eta$ by hypothesis.

If $\ker \eta/M = G_k(\alpha,c)/M$, then $G_k(\alpha,c)\eta \leq M$, and $(\gamma_c(G_k(\alpha,c)))\eta = \{1\}$, since $\gamma_c(G_k(\alpha,c))$ has exponent $p$ and $\gamma_{c+1}(G_k(\alpha,c)) = \{1\}$, so again $S$ admits $\eta$.

Now consider the case $\{1\} < \ker \eta/M < G_k(\alpha,c)/M$. Let $\{g_1,\ldots,g_k\}$ be a free generating set for $G_k(\alpha,c)$ and let the dimension of the space generated by $\{g_i\eta M;\ i = 1,\ldots,k\}$ be $r$, where $1 \leq r < k$. Then $r$ of the $g_i\eta M$ are linearly independent, say $g_{i_1}\eta M,\ldots,g_{i_r}\eta M$, where $\{i_1,\ldots,i_r\} \subseteq \{1,\ldots,k\}$. Then we can write

$$g_j\eta = g_{i_1}^{n_{j1}} \cdots g_{i_r}^{n_{jr}} \quad \text{modulo } M,$$

where $0 \leq n_{ji} < p$, $i \in \{1,\ldots,r\}$, $j \in \{1,\ldots,k\}$.

Define an automorphism $\phi$ of $G_k(\alpha,c)$ as follows :

$$g_j\phi = \begin{cases} g_i & \text{for } j \in \{i_1,\ldots,i_r\} \\ g_j(g_j\eta)^{-1} & \text{otherwise.} \end{cases}$$

Then

$$g_j\phi\eta = \begin{cases} g_j\eta & \text{for } j \in \{i_1,\ldots,i_r\} \\ 1 \bmod M & \text{otherwise.} \end{cases}$$

Define another automorphism $\hat{\eta}$ of $G_k(\alpha,c)$ as follows :

$$g_j\hat{\eta} = \begin{cases} g_j\phi\eta & \text{for } j \in \{i_1,\ldots,i_r\} \\ f_j & \text{otherwise} \end{cases}$$

where the $f_j$ are chosen in such a way that $\{g_j\hat{\eta}M;\ j = 1,\ldots,k\}$ forms as basis for $F/M$.

Now $S$ admits $\hat{\eta}$ since it is an automorphism of $G_k(\alpha,c)$. Since $S$ admits the deletions of $G_k(\alpha,c)$, it is generated by homogeneous elements. If $x \in S$ is homogeneous then $x\phi\eta = 1$, or $x\phi\eta = x\hat{\eta}$, and so $S$ admits $\phi\eta$. But $\phi \in \mathrm{Aut}\, G_k(\alpha,c)$, so $S\phi \subseteq S$, and in fact $S\phi = S$. Thus we have shown that $S \supseteq S\phi\eta = S\eta$, which is the required result.

To make this theorem useful we now look for a small set, $P$, of automorphisms of $G_k(\alpha,c)$ with the property that closure under $P$ implies closure under the automorphism group of $G_k(\alpha,c)$. To this end we define the automorphisms of $G_k(\alpha,c)$, $\beta$, $\rho_j$, $2 \le j \le k$, and $\varepsilon(a)$, where $a$ generates $GF(p)$, as follows :

$$g_i\beta = \begin{cases} g_1g_2 & i = 1 \\ g_i & 2 \le i \le k, \end{cases}$$

$$g_i\rho_j = \begin{cases} g_j & i = 1 \\ g_1 & i = j \\ g_i & i \ne j,\ 2 \le i \le k \end{cases}$$

and

$$g_i\varepsilon(a) = \begin{cases} g_i & 1 \le i \le k-1 \\ g_k{}^a & i = k. \end{cases}$$

Let $P = \{\beta;\ \rho_j,\ 2 \le j \le k;\ \varepsilon(a)\}$.

We have noted that $\mathrm{Aut}\,(G_k(\alpha,c)/M) \cong GL(k,p)$. We make use of this to show that $P$ is the required set, but first we introduce some elements of $GL(k,p)$.

For any $i \neq j$ and any $\lambda \in GF(p)$, we denote by $B_{ij}(\lambda)$ the matrix obtained from the unit matrix by replacing the element $a_{ij} = 0$ of the unit matrix by $\lambda$. For any $j \in \{2,\ldots,k\}$ we denote by $P_j$ the matrix obtained from the unit matrix by replacing the elements $a_{1j} = a_{j1} = 0$ by $1$ and by replacing the elements $a_{11} = a_{jj} = 1$ of the unit matrix by $0$. Let $D(\mu)$ be the diagonal matrix with diagonal entries $\{1,\ldots,1,\mu\}$, where $\mu \in GF(p)$.

We use the following theorem (4.1 of [ 1]) without proof.

3.1.4 Theorem : Every non-singular matrix $A$ over $GF(p)$ can be written in the form $B.D(\mu)$, where $B$ is a product of matrices $B_{ij}(\lambda)$, and $\mu \in GF(p)$, $\mu \neq 0$.

Thus we have a generating set for $GL(k,p)$, but it is not yet in the form we require. We use the following well known result to get 3.1.6.

3.1.5 Lemma : The matrices $B_{ij}(\lambda)$ are contained in the group generated by $\{B_{12}(\lambda);\ P_j,\ 2 \leq j \leq k\}$.

Proof : We first note that $B_{ij}(\lambda) = (B_{ij}(1))^{\lambda}$, so it is sufficient to show that the matrices $B_{ij}(1)$ are in the group generated by $\{B_{12}(\lambda);\ P_j,\ 2 \leq j \leq k\}$.

An easy calculation gives the following results.

$$B_{i2}(1) = P_i\, B_{12}(1)\, P_i \qquad \text{for } 3 \leq i \leq k,$$

$$B_{i1}(1) = P_2\, B_{i2}(1)\, P_2 \qquad \text{for } 3 \leq i \leq k,$$

$$B_{1j}(1) = P_j\, B_{j1}(1)\, P_j \qquad \text{for } 3 \leq j \leq k,$$

$$B_{ij}(1) = P_i\, B_{1j}(1)\, P_i \qquad \text{for } 2 \leq i \leq k,\ 3 \leq j \leq k,$$

$$B_{21}(1) = P_2\, B_{12}(1)\, P_2.$$

3.1.6 Corollary : For $p \neq 2$, $GL(k,p)$ is generated by $\{B_{12}(1);\ P_j,\ 2 \leq j \leq k;\ D(a)\}$ where $a$ generates the multiplicative group of $GF(p)$. $GL(k,2)$ is generated by $\{B_{12}(1);\ P_j,\ 2 \leq j \leq k\}$.

We now have a generating set for $GL(k,p)$, and we can show that $P$ has the property mentioned earlier. First we prove a preliminary result.

3.1.7 Lemma : Let $\eta_1$ and $\eta_2$ be endomorphisms of $G_k(\alpha,c)$ such that $\eta_1/M = \eta_2/M$. Then $\eta_1\big|_{\gamma_c(G_k(\alpha,c))} = \eta_2\big|_{\gamma_c(G_k(\alpha,c))}$.

Proof : The result follows because $\gamma_c(G_k(\alpha,c))$ has exponent $p$ and $\gamma_{c+1}(G_k(\alpha,c)) = 1$ .

3.1.8 Theorem : Let $S$ be a subgroup of $\gamma_c(G_k(\alpha,c))$ that admits the members of $P$ and deletions of $G_k(\alpha,c)$. Then $S$ is fully invariant in $G_k(\alpha,c)$.

Proof : It is sufficient to show that $S$ admits the automorphisms of $G_k(\alpha,c)$.

$G_k(\alpha,c)/M$ may be regarded as a $k$-dimensional vector space over $GF(p)$. So with respect to a suitable basis of $G_k(\alpha,c)/M$, $\beta/M$ is the linear transformation whose matrix is $B_{12}(1)$, $\rho_j/M$ is the linear transformation whose matrix is $P_j$, where $2 \leq j \leq k$, and $\varepsilon(a)$ is the linear transformation whose matrix is $D(a)$. From 3.1.6 then, $P/M = \{\beta/M;\ \rho_j/M,\ 2 \leq j \leq k;\ \varepsilon(a)/M\}$ generates the automorphism group of $G_k(\alpha,c)/M$.

Let $\eta$ be any automorphism of $G_k(\alpha,c)$. Then $\eta/M$ is an automorphism of $G_k(\alpha,c)$ and we can write $\eta/M$ as a word in the generators, say

$$\eta/M = h(\beta/M,\ \rho_j/M,\ \varepsilon(a)/M).$$

Put $\nu = h(\beta, \rho_j, \varepsilon(a))$.

Then $\nu/M = \eta/M$, and since $S$ admits $\nu$, it follows from 3.1.7 that $S$ admits $\eta$.

## 3.2 A Non-Distributive Lattice

We now consider $\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_6$. Let $\{g_1, g_2, g_3\}$ be a free generating set for $G = F_3(\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_6)$. $\gamma_6(G)$ is an elementary two-group, since it is a subgroup of $G'$. A basis for $\gamma_6(G)$ is listed in table 1 of the appendix. The commutators listed are obviously in $\gamma_6(G)$. That they are a basis follows from 1.1.7 .

From 3.1.8 to show that a subgroup of $\gamma_6(G)$ is fully invariant in $G$ we have only to show that it is closed under the automorphisms $\beta$, $\rho_2$ and $\rho_3$, and the deletions of $G$. We now define certain subgroups of $\gamma_6(G)$ in terms of their generators.

Let

$$w_1 = c_5c_{14},\quad w_2 = c_1c_7c_{11},\quad w_3 = c_3c_{10}c_{16},\quad w_4 = c_2c_8c_{12}c_{15}$$
$$w_5 = c_3c_5c_{12}c_{13},\quad w_6 = c_2c_4c_{11}c_{14},\quad w_7 = c_{25},\quad w_8 = c_{22},$$
$$w_9 = c_{19},\quad w_{10} = c_2c_5c_7,\quad w_{11} = c_{12}c_{14}c_{16},\quad w_{12} = c_{24}c_{25},$$
$$w_{13} = c_{21}c_{22},\quad w_{14} = c_{19}c_{20},\quad w_{15} = c_{22}c_{23},\quad w_{16} = c_{18}c_{19},$$
$$w_{17} = c_{25}c_{26},\quad u_1 = c_5,\quad u_2 = c_7,\quad u_3 = c_3c_{10},$$
$$v_1 = c_5c_9,\quad v_2 = c_6c_7,\quad v_3 = c_3c_{10}c_{17}.$$

Now put
$$V = \mathrm{gp}\,\{w_1,\ldots,w_{17}\},$$
$$D_1 = \mathrm{gp}\,\{u_1,u_2,u_3,V\}$$
$$D_2 = \mathrm{gp}\,\{v_1,v_2,v_3,V\},$$
and
$$C = \mathrm{gp}\,\{u_1v_1,u_2v_2,u_3v_3,V\}.$$

It can be shown using 3.1.8, that $V$ is a fully invariant subgroup of $G$, and it follows from table 2 in the appendix that

$D_1$, $D_2$ and $C$ are fully invariant subgroups of $G$, if $V$ is fully invariant. It can also be seen that $C \cap D_1 = V$, $C \cap D_2 = V$ and $C \cap D_1D_2 = C$, which with 3.1.1 gives the required result.

We also give an example to show that closure of a subgroup $S$ under the automorphisms is not sufficient to imply that $S$ is fully invariant.

Let $S = \text{gp}\,\{w_1, w_2, w_3, w_4w_7, w_5w_8, w_6w_9\}$, Then it can be seen from table 3 in the appendix that $S$ is closed under the automorphisms $\beta$, $\rho_2$ and $\rho_3$, and it is therefore closed under the automorphism group of $G$. But $S$ is not closed under the deletions and is therefore not fully invariant.

The first example of non-distributivity I found in lat $(\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_6)$ was in the verbal subgroup lattice of the free group of rank four, and I am indebted to Dr M.F. Newman for his help in finding this example of non-distributivity in the free group of rank three. This result, however, is not necessarily the best possible, for the question of the distributivity of the lattices of $\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_4$ and $\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_5$ still remains open.

# CHAPTER 4

## A Basis Theorem

Throughout this chapter we will be working in the variety $\underline{\underline{A}}_p\underline{\underline{T}}_p$ where for $p \neq 2$, $\underline{\underline{T}}_p = \underline{\underline{B}}_p \wedge \underline{\underline{N}}_2$ and $\underline{\underline{T}}_2 = \underline{\underline{B}}_4 \wedge \underline{\underline{N}}_2$, and we let $G(p)$ represents $F_\infty(\underline{\underline{A}}_p\underline{\underline{T}}_p)$. We have shown in chapter 1 that $\underline{\underline{T}}_p(G(p))$ is an elementary abelian $p$-group, and in this chapter we find an explicit basis for $\underline{\underline{T}}_p(G(p))$.

In section 4.1 we introduce some further notation and give a formal statement of the main result. The proof of this result modulo three principal lemmas is given in 4.2, while the proofs of these lemmas occupy 4.3 and 4.4.

Much of this chapter, and chapter 5 is closely modelled on the work of M.S. Brooks ([6] and [8]) on metabelian varieties. The details and the results obtained are quite different from his, but the underlying philosophy of chapters four and five is that of Brooks.

### 4.1 Statement of the Main Theorem

Before stating our main result for this chapter we introduce some more notation for the set of commutators of a group.

4.1.1 Notation: Let $H$ be any group and let $\underline{\underline{c}}$ be an ordered subset of $C(H)$. We define another ordered subset $\underline{\underline{a}}(\underline{\underline{c}})$ of $C(H)$ by

$$\underline{\underline{a}}(\underline{\underline{c}}) = \{(c_1,c_2) : c_1,c_2 \in \underline{\underline{c}},\ c_1 > c_2\}$$

where $(c_1,c_2) < (d_1,d_2)$ if $c_1 < d_1$, or $c_1 = d_1$ and $c_2 < d_2$.

At this stage it is necessary, for any group $H$, to extend our set of commutators to include expressions of the form $c^n$, where $c \in C(H)$ and $n \in I^+$. These are to be unique as written and take the obvious values in H, namely, $[c^n] = [c]^n$.

Then for $\underline{\underline{c}}$ as above, and $n \in I^+$, we define $(\underline{\underline{c}})^n$ by

$$(\underline{\underline{c}})^n = \{c^n : c \in \underline{\underline{c}}\}$$

where $c_1^n < c_2^n$ if, and only if $c_1 < c_2$.

With this notation we now define some subsets of $C(G(p))$. Let $\underline{\underline{g}}(p) = \{g_{pi} : i \in I^+\}$ be a free generating set for $G(p)$ which is well-ordered by its indexing set.

For $p \neq 2$, let $Q(p)$ be defined by

$$Q(p) = \{(a,\alpha_1 c_1,\ldots,\alpha_k c_k) : a \in (B(\underline{\underline{g}}(p);p)\backslash \underline{\underline{a}}(\underline{\underline{g}}(p))) \cup (\underline{\underline{g}}(p))^p,$$
$$c_i \in \underline{\underline{a}}(\underline{\underline{g}}(p)), c_1 < c_2 < \ldots < c_k,\ 1 \leq \alpha_i \leq p-1,\ k \in I^+\},$$

and let $Q(2)$ be defined by

$$Q(2) = \{(a,\alpha_1 c_1,\ldots,\alpha_k c_k) : a \in (B(\underline{\underline{g}}(2);2)\backslash \underline{\underline{a}}(\underline{\underline{g}}(2))),$$
$$c_i \in \underline{\underline{a}}(\underline{\underline{g}}(2)), c_1 < \ldots < c_k, \alpha_i = 1,\ i = 1,\ldots,k\}.$$

For $p \neq 2$, put $\underline{\underline{b}}(p) = \underline{\underline{a}}(\underline{\underline{g}}(p))$ and put $\underline{\underline{b}}(2) = \underline{\underline{a}}(\underline{\underline{g}}(2)) \cup (\underline{\underline{g}}(2))^2$ where the order on $\underline{\underline{b}}(2)$ is determined by the orders on $\underline{\underline{a}}(\underline{\underline{g}}(2))$ and $(\underline{\underline{g}}(2))^2$ and the condition that if $b_1 \in \underline{\underline{a}}(\underline{\underline{g}}(2))$, $b_1 < b_2$ for all $b_2 \in (\underline{\underline{g}}(2))^2$.

We now define the subsets $R(p)$ of $C(G(p))$. For $p \neq 2$,

$$R(p) = (\underline{\underline{g}}(p))^p \cup (\underline{\underline{a}}(\underline{\underline{g}}(p)))^p \cup (B(\underline{\underline{g}}(p);p)\backslash \underline{\underline{a}}(\underline{\underline{g}}(p))) \cup B(\underline{\underline{b}}(p);p) \cup Q(p),$$

and

$$R(2) = (\underline{\underline{g}}(2))^4 \cup (\underline{\underline{a}}(\underline{\underline{g}}(2)))^2 \cup (B(\underline{\underline{g}}(2);2)\backslash \underline{\underline{a}}(\underline{\underline{g}}(2))) \cup B(\underline{\underline{b}}(2);2) \cup Q(2).$$

We state our main theorem as follows:

4.1.2 <u>Theorem</u>: $\underline{\underline{T}}_p(G(p))$ is a free abelian group of exponent p. The valuation mapping $\phi(p) : R(p) \to G(p)$ is one-to-one, and $R(p)\phi(p)$ is a basis for $\underline{\underline{T}}_p(G(p))$.

## 4.2 Skeletal Proof of 4.1.2

Most of the proof of 4.1.2 will be carried out in finitely generated subgroups of $G(p)$. For any integer $r > 1$ let $\underline{\underline{g}}_r(p) = \{g_{p1},\dots,g_{pr}\} \leq \underline{\underline{g}}(p)$, and let $G_r(p) = gp(\underline{\underline{g}}_r(p))$. Let $\underline{\underline{b}}_r(p) = \underline{\underline{b}}(p) \cap C(G_r(p))$, $Q_r(p) = Q(p) \cap C(G_r(p))$ and $R_r(p) = R(p) \cap C(G_r(p))$.

In this section it is shown how 4.1.2 is deduced from the following three lemmas. The proof of these lemmas is postponed until sections 4.3 and 4.4.

4.2.1 <u>Lemma</u>: For all $r \geq 2$, $p$ an odd prime, $\underline{\underline{T}}_p(G_r(p))$ is a free abelian group of exponent $p$ and rank $(r-1)p^{\frac{1}{2}r(r+1)} + 1$. $\underline{\underline{T}}_2(G_r(2))$ is free abelian of exponent 2 and rank $(r-1)2^{\frac{1}{2}r(r+3)} + 1$.

4.2.2 <u>Lemma</u>: For all $r \geq 2$, $p$ an odd prime, $|R_r(p)| = (r-1)p^{\frac{1}{2}r(r+1)} + 1$. $|R_r(2)| = (r-1)2^{\frac{1}{2}r(r+3)} + 1$.

4.2.3 <u>Lemma</u>: For all $r \geq 2$,

$$\underline{\underline{T}}_p(G_r(p)) = gp\{R_r(p)\phi(p)\}.$$

Here, the rank of $\underline{\underline{T}}_p(G_r(p))$ and the cardinality of $R_r(p)$ are not important in themselves; only their equality is required, and this is used to prove the following result.

4.2.4 <u>Lemma</u>: For any integer $r \geq 2$ the valuation mapping $\phi(p)|_{R_r(p)} : R_r(p) \to G_r(p)$ is one-to-one, and $R_r(p)\phi(p)$ is a basis for $\underline{\underline{T}}_p(G_r(p))$.

<u>Proof</u>: From 4.2.2 $|R_r(p)\phi(p)| \leq (r-1)p^{\frac{1}{2}r(r+1)} + 1$ for $p \neq 2$, and $|R_r(2)\phi(2)| \leq (r-1)2^{\frac{1}{2}r(r+3)}+1$, and equality holds only if $\phi(p)|_{R_r(p)}$

is one-to-one. On the other hand, since from 4.2.3 $R_r(p)\phi(p)$ is a generating set for $\underline{\underline{T}}_p(G_r(p))$, it follows from 4.2.1 that $|R_r(p)\phi(p)| \geq (r-1)p^{\frac{1}{2}r(r+1)} + 1$ for $p \neq 2$, and $|R_r(2)\phi(2)| \geq (r-1)2^{\frac{1}{2}r(r+3)} + 1$, and in all cases equality holds only if $R_r(p)\phi(p)$ is a basis for $\underline{\underline{T}}_p(G_r(p))$.

<u>Proof of 4.1.2</u>: That $\underline{\underline{T}}_p(G(p))$ is free abelian of exponent $p$ follows directly from 1.3.2.

The mapping $\phi(p) : R(p) \to G(p)$ is one-to-one because any two distinct elements belonging to $R(p)$ are also members of $R_r(p)$ for sufficiently large $r$, and therefore have distinct values, since $\phi(p)|_{R_r(p)}$ is one-to-one by 4.2.4.

$R(p)\phi(p)$ generates $\underline{\underline{T}}_p(G(p))$ because any element $w$ in $\underline{\underline{T}}_p(G(p))$ is also a member of $\underline{\underline{T}}_p(G_r(p))$ for large enough $r$, and by 4.2.3 $\underline{\underline{T}}_p(G_r(p)) = gp\{R_r(p)\phi(p)\} \subseteq gp\{R(p)\phi(p)\}$.

To verify that $R(p)\phi(p)$ is a basis for $\underline{\underline{T}}_p(G(p))$ it remains to show that no non-trivial relation exists between its members. But any non-trivial relation involving the members of $R(p)\phi(p)$ would provide an example of a non-trivial relation among the members of $R_r(p)\phi(p)$, and this would contradict 4.2.4.

## 4.3 <u>The Proofs of 4.2.1 and 4.2.2.</u>

We will need the following observation:

4.3.1 <u>Lemma</u>: For any prime $p \neq 2$, $|F_r(\underline{\underline{T}}_p)| = p^{\frac{r(r+1)}{2}}$ $|F_r(\underline{\underline{T}}_2)| = 2^{\frac{1}{2}r(r+3)}$.

<u>Proof</u>: $F_r(\underline{\underline{T}}_p)/F_r'(\underline{\underline{T}}_p)$ is a free abelian group of exponent $p$ and rank $r$. For $p = 2$, the exponent is $4$. So for $p \neq 2$,

$$\left|\frac{F_r(\underline{\underline{T}}_p)}{F_r'(\underline{\underline{T}}_p)}\right| = p^r, \text{ and } |F_r(\underline{\underline{T}}_2)/F_r'(\underline{\underline{T}}_2)| = 4^r = 2^{2r}.$$

$F_r'(\underline{\underline{T}}_p)$ is also free abelian of exponent $p$ and rank $\frac{1}{2}r(r-1)$.
So $|F_r'(\underline{\underline{T}}_p)| = p^{\frac{1}{2}r(r-1)}$.

Therefore we have

$$\begin{aligned}
|F_r(\underline{\underline{T}}_p)| &= |F_r(\underline{\underline{T}}_p)/F_r'(\underline{\underline{T}}_p)|.|F_r'(\underline{\underline{T}}_p)| \\
&= p^r \,.\, p^{\frac{1}{2}r(r-1)} \qquad \text{for } p \neq 2 \\
&= p^{\frac{1}{2}r(r+1)},
\end{aligned}$$

and

$$\begin{aligned}
|F_r(\underline{\underline{T}}_2)| &= 2^{2r} \,.\, 2^{\frac{1}{2}r(r-1)} \\
&= 2^{\frac{1}{2}r(r+3)}.
\end{aligned}$$

<u>Proof of 4.2.1</u>: $G_r(p) = F_r(\underline{\underline{A}}_p\underline{\underline{T}}_p) \cong F_r/\underline{\underline{A}}_p\underline{\underline{T}}_p(F_r)$, where $F_r$ is absolutely free on $r$ generators. Therefore $\underline{\underline{T}}_p(G_r(p)) = \underline{\underline{T}}_p(F_r)/\underline{\underline{A}}_p(\underline{\underline{T}}_p(F_r))$, where $\underline{\underline{T}}_p(F_r) \leq F_r$ is free of rank $(r-1)|\underline{\underline{T}}_p(F_r)|+1$ using Schreier's formula. It follows immediately from the expression above that $\underline{\underline{T}}_p(G_r(p))$ is free abelian of exponent $p$, and rank $(r-1)p^{\frac{1}{2}r(r+1)}+1$ for $p\neq 2$, and rank $(r-1)2^{\frac{1}{2}r(r+3)}+1$ for $p = 2$, using 4.3.1.

<u>Proof of 4.2.2</u>: We note that $R_r(p)$ is a disjoint union of its subsets, $(\underline{\underline{g}}_r(p))^p$, $(\underline{\underline{a}}(\underline{\underline{g}}_r(p)))^p$, $B(\underline{\underline{g}}_r(p);p)\backslash\underline{\underline{a}}(\underline{\underline{g}}_r(p))$, $B(\underline{\underline{b}}_r(p);p)$ and $Q_r(p)$ for $p \neq 2$, and $(\underline{\underline{g}}_r(2))^4$, $(\underline{\underline{a}}(\underline{\underline{g}}_r(2)))^2$, $B(\underline{\underline{g}}_r(2);2)\backslash\underline{\underline{a}}(\underline{\underline{g}}_r(2))$, $B(\underline{\underline{b}}_r(2);2)$ and $Q_r(2)$ for $p = 2$, so that the cardinality of $R_r(p)$ is just the sum of the cardinalities of these subsets, and these are determined as follows:

$|(\underline{\underline{g}}_r(p))^p| = r$, $|(\underline{\underline{a}}(\underline{\underline{g}}_r(p)))^p| = \frac{1}{2}r(r-1)$, $|\underline{\underline{g}}_r(p)| = r$ so that $|B(\underline{\underline{g}}_r(p);p)| = (r-1)(p^r-1)$ and $|B(\underline{\underline{g}}_r(p);p)\backslash\underline{\underline{a}}(\underline{\underline{g}}_r(p))| = (r-1)(p^r-1) - \frac{1}{2}r(r-1)$. For $p \neq 2$ $|\underline{\underline{b}}_r(p)| = \frac{1}{2}r(r-1)$ so that $|B(\underline{\underline{b}}_r(p);p)| = (\frac{1}{2}r(r-1)-1)(p^{\frac{1}{2}r(r-1)} - 1)$, and $|Q_r(p)| = ((r-1)(p^r-1) - \frac{1}{2}r(r-1) + r)(p^{\frac{1}{2}r(r+1)} - 1)$.

Thus for $p \neq 2$ we have

$$|R_r(p)| = r + \tfrac{1}{2}r(r-1) + (r-1)(p^r-1) - \tfrac{1}{2}r(r-1) + (\tfrac{1}{2}r(r-1) - 1)(p^{\frac{1}{2}r(r-1)}) - 1)$$
$$+ ((r-1)(p^r-1) - \tfrac{1}{2}r(r-1) + r)(p^{\frac{1}{2}r(r-1)} - 1)$$
$$= (r-1)p^{\frac{1}{2}r(r+1)} + 1.$$

For $p = 2$ we have $|(\underline{\underline{g}}_r(2))^4| = r$, $|(\underline{\underline{a}}(\underline{\underline{g}}_r(2)))^2| = \tfrac{1}{2}r(r-1)$,
$|B(\underline{\underline{g}}_r(2);2)| = (r-1)(2^r-1)$ and hence
$|B(\underline{\underline{g}}_r(2);2)\backslash\underline{\underline{a}}(\underline{\underline{g}}_r(2))| = (r-1)(2^r-1) - \tfrac{1}{2}r(r-1)$, $|\underline{\underline{b}}_r(2)| = \tfrac{1}{2}r(r+1)$
so that $|B(\underline{\underline{b}}_r(2);2| = (\tfrac{1}{2}r(r+1) - 1)(2^{\frac{1}{2}r(r+1)} - 1)$, and
$|Q_r(2)| = ((r-1)(2^r-1) - \tfrac{1}{2}r(r-1))(2^{\frac{1}{2}r(r+1)} - 1)$.

Hence we have

$$|R_r(2)| = r + \tfrac{1}{2}r(r-1) + (r-1)(2^r-1) - \tfrac{1}{2}r(r-1) + (\tfrac{1}{2}r(r+1) - 1)(2^{\frac{1}{2}r(r+1)} - 1)$$
$$+ ((r-1)(2^r-1) - \tfrac{1}{2}r(r-1))(2^{\frac{1}{2}r(r+1)} - 1)$$
$$= (r-1)2^{\frac{1}{2}r(r+3)} + 1.$$

## 4.4 Proof of 4.2.3

The proof of 4.2.3 requires a number of preliminary results. A generating set for $\underline{\underline{AA}}_p(G_r(p))$ is found and this is enlarged to make a generating set for $\underline{\underline{T}}_p(G_r(p))$. Our first step is to find a generating set for $\underline{\underline{A}}_p(G_r(p))$. For simplicity of notation we let $\underline{\underline{g}}_r(p) = \{g_1,\ldots,g_r\}$.

4.4.1 Lemma: Let $u \in \underline{\underline{A}}_p(G_r(p))$. Then $u = (g_1^p)^{\alpha_1}\ldots(g_r^p)^{\alpha_r}v$, where $0 \leq \alpha_i < p$ for $p \neq 2$, and $0 \leq \alpha_i < 4$ for $p = 2$, and $v \in G_r'(p)$.

Proof: $u = v_1^p\ldots v_s^p$ for some $v_1,\ldots,v_s \in G_r(p)$
$= (v_1\ldots v_s)^p \bmod G_r'(p)$
$= w^p \bmod G_r'(p)$ where $w = v_1\ldots v_s$ .

We can write $w = g_{i_1}^{\varepsilon_1}\ldots g_{i_t}^{\varepsilon_t}$, where $i_j \neq i_{j+1}$, and $i_j \in \{1,\ldots,r\}$,

$0 < \varepsilon_j < p^2$ for $p \neq 2$, $0 \leq \varepsilon_j < 8$ for $p = 2$. Then

$$\begin{aligned} w^p &= (g_{i_1}^{\varepsilon_1} \ldots g_{i_t}^{\varepsilon_t})^p \\ &= (g_{i_1}^{\varepsilon_1})^p \ldots (g_{i_t}^{\varepsilon_t})^p \mod G_r'(p) \\ &= (g_1^p)^{\alpha_1} \ldots (g_r^p)^{\alpha_r} \mod G_r'(p) \end{aligned}$$

where $0 \leq \alpha_i < p$ for $p \neq 2$, and $0 \leq \alpha_i < 4$ for $p = 2$.

In the next lemma we look at $G_r'(p)$ modulo $\underline{\underline{AA}}_p(G_r'(p))$. Here we rely on the result of Brooks stated in 1.1.7 concerning a basis for the derived group of $F_r(\underline{\underline{AA}}_p)$.

4.4.2 <u>Lemma</u>: Let $v \in G_r'(p)$. Then $v = b_1^{e_1} \ldots b_s^{e_s} t$, where for $i \in \{1,\ldots,s\}$, $b_i \in B(\underline{\underline{g}}_r(p);p)\phi(p)$, and $t \in \underline{\underline{AA}}_p(G_r(p))$.

<u>Proof</u>: $G_r(p)/\underline{\underline{AA}}_p(G_r(p))$ is a homomorphic image of $F_r(\underline{\underline{AA}}_p)$ under the homomorphism $\theta : F_r(\underline{\underline{AA}}_p) \to G_r(p)/\underline{\underline{AA}}_p(G_r(p))$ given by $f_j\theta = g_j\underline{\underline{AA}}_p(G_r(p))$, $j = 1,\ldots,r$, where $\underline{\underline{f}}_r = \{f_1,\ldots,f_r\}$ is a free generating set for $F_r(\underline{\underline{AA}}_p)$.

So for any $v \in G_r'(p)/\underline{\underline{AA}}_p(G_r(p))$, there is a $u \in F_r'(\underline{\underline{AA}}_p)$ such that $u$ is a product of distinct elements of $B(\underline{\underline{f}}_r(p);p)$ evaluated in $F_r(\underline{\underline{AA}}_p)$. Then $v = u\theta = b_1^{e_1} \ldots b_s^{e_s}\, \underline{\underline{AA}}_p(G_r(p))$ where $b_i \in B(\underline{\underline{g}}_r(p);p)\phi(p)$.

We combine these last two results to get the following result:

4.4.3 <u>Lemma</u>: Let $u \in \underline{\underline{A}}_p(G_r(p))$. Then $u = (g_1^p)^{\alpha_1} \ldots (g_r^p)^{\alpha_r} b_1^{e_1} \ldots b_s^{e_s} t$, where $0 \leq \alpha_i < p$ ($0 \leq \alpha_i < 4$ for $p = 2$), $0 \leq e_i < p^2$, $b_i \in B(\underline{\underline{g}}_r(p);p)\phi(p)$, for $i \in \{1,\ldots,s\}$ and $t \in \underline{\underline{AA}}_p(G_r(p))$.

4.4.4 <u>Corollary</u>: $\underline{\underline{A}}_p(G_r(p))$ is generated by $S_r(p)\phi(p)$, where $S_r(p) = (\underline{\underline{g}}_r(p))^p \cup B(\underline{\underline{g}}_r(p);p)$.

Proof: $\underset{=}{A}_p(G_r(p))$ is a nilpotent group, and so $\underset{==}{AA}_p(G_r(p))$ is contained in the Frattini subgroup of $\underset{=}{A}_p(G_r(p))$. Hence by 31.25 of [17], since $\underset{=}{A}_p(G_r(p))$ is generated by $S_r(p)\phi(p)$ modulo $\underset{==}{AA}_p(G_r(p))$, $\underset{=}{A}_p(G_r(p))$ is generated by $S_r(p)\phi(p)$.

The following theorem appears as 5.4 of [16]. We state it without proof.

4.4.5 Theorem: Let the group $H$ be generated by $a_1,\ldots,a_r$. Then $H_n/H_{n+1}$ is abelian and is generated by cosets of the simple n-fold commutator-elements, $[a_{\zeta_1},\ldots,a_{\zeta_n}]$, where $\zeta_i \in \{1,2,\ldots,r\}$.

In terms of the groups we have been using we have the following result.

4.4.6 Lemma: $\underset{=}{N}_\alpha \underset{=}{A}_p(G_r(p))/\underset{=}{N}_{\alpha+1}\underset{=}{A}_p(G_r(p))$ is generated by the cosets of the commutator-elements $[u,c_1,c_2\ldots,c_\alpha]$ where $u \in S(p)\phi(p)$ and $c_i \in \underset{=}{b}_r(p)\phi(p)$, $i \in \{1,2,\ldots,\alpha\}$.

Proof: It follows from 4.4.5 that $\underset{=}{N}_\alpha \underset{=}{A}_p(G_r(p))/\underset{=}{N}_{\alpha+1}\underset{=}{A}_p(G_r(p))$ is generated by the cosets of the commutator elements $[s_{i_1},\ldots,s_{i_{\alpha+1}}]$ where $s_{i_j} \in S(p)\phi(p)$. However, only those of the form $[u,c_1,\ldots,c_\alpha]$ and $[c_1,u,c_2,\ldots,c_\alpha]$, where $u \in S(p)\phi(p)$ and $c_i \in \underset{=}{b}_r(p)\phi(p)$ are non-trivial. The result follows by noting that $[c_1,u,c_2\ldots,c_\alpha] = [u,c_1,c_2,\ldots,c_\alpha]^{-1}$.

4.4.7 Corollary: $\underset{=}{N}_\alpha \underset{=}{A}_p(G_r(p))$ is generated by the elements $[u,c_{i_1},\ldots,c_{i_s}]$, where $u \in S(p)\phi(p)$, $c_{i_j} \in \underset{=}{b}_r(p)\phi(p)$ for $j \in \{1,\ldots,s\}$ and $\alpha \le s \le \beta$ for some $\beta \in I^+$.

Proof: $\underset{=}{A}_p(G_r(p))$ is nilpotent of class $\beta$ for some $\beta \in I^+$. Hence $\underset{=}{N}_\beta \underset{=}{A}_p(G_r(p)) = \{1\}$, and the result follows immediately from 4.4.6.

We use these results to find a generating set for $\underset{==}{AA}_p(G_r(p))$.

4.4.8 Theorem: $\underline{\underline{AA}}_p(G_r(p))$ is generated by $(B(\underline{\underline{b}}_r(p);p) \cup P_r(p))\phi(p)$ where $\phi(p)$ is the valuation map $\phi(p) : R_r(p) \to G_r(p)$.

<u>Proof</u>: We have shown that $\underline{\underline{AA}}_p(G_r(p))$ is generated by the set $A(p)\phi(p)$ where $A(p) = \{(u,c_{i_1},\ldots,c_{i_s});\ 1 \leq s < \beta,\ u \in S(p),\ c_{i_j} \in \underline{\underline{b}}_r(p)\}$ and where $S(p) = (\underline{\underline{g}}_r(p))^p \cup B(\underline{\underline{g}}_r(p);p)$, and it is immediate that $A(p)\phi(p) \supseteq (B(\underline{\underline{b}}_r(p);p) \cup P_r(p))\phi(p)$. We will show that given $a \in A$ such that $a \notin B(\underline{\underline{b}}_r(p);p) \cup P_r(p)$ then $a$ is trivial, or there is an $a_1 \in B(\underline{\underline{b}}_r(p);p) \cup P_r(p)$ such that $[a] = [a_1]$, or $[a]$ can be written as a product of elements in $(B(\underline{\underline{b}}_r(p);p) \cup P_r(p))\phi(p)$.

Let $a = (u,c_{i_1},\ldots,c_{i_s})$, where $u \in S(p)\backslash\underline{\underline{b}}_r(p)$. By we can write $[a] = [[u],\alpha_1[c_1],\ldots,\alpha_k[c_k]]$ where $k = \underline{\underline{b}}_r(p)$ and $\alpha_i \geq 0$. If $\alpha_i \geq p$ for some $i \in \{1,\ldots,k\}$, then by 1.3.4 $[a] = 1$. Otherwise $[a]$ is nontrivial and $[a] = [a_1]$ where $a_1 = (u,\alpha_1 c_1,\ldots,\alpha_k c_k)$, $0 \leq \alpha_i < p$ and $a_1 \in P_r(p)$.

Now consider the case $a = (u,c_{i_1},\ldots,c_{i_s})$ where $u \in \underline{\underline{b}}_r(p)$. Then we can write

$$\begin{aligned}
[a] &= [[c_a],[c_{i_1}],\ldots,[c_{i_s}]] \\
&= [[c_a],[c_{i_1}],\alpha_1[c_1],\ldots,\alpha_k[c_k]], \quad \text{where } \alpha_i \geq 0 \text{ by 1.2.3} \\
&= [[c_a],[c_{i_1}],\delta] \quad \text{for some degree function } \delta \text{ on } \underline{\underline{b}}_r(p).
\end{aligned}$$

If $\alpha_i \geq p$ for some $i \in \{1,\ldots,k\}$, then $[a] = 1$ by 1.3.5. So we assume $\alpha_i < p$ for $i \in \{1,\ldots,k\}$.

If $c_i$ is not min supp $\delta$, then by 1.2.3 we can write $[a] = [[c_a],[c_s],\delta][[c_{i_1}],[c_2],\delta]^{-1}$, where $c_s$ = min supp . (If $c_a = c_s$, then $[[c_a],[c_s],\delta] = 1$.) So we have now that $[a] = [[c_i],[c_j],\delta]$ where $c_j$ = min supp $\delta$, or $[a]$ can be written as a product of such commutator elements. So we now consider $[a] = [[c_i],[c_j],\delta]$ where $c_j$ = min supp $\delta$. If $\delta(c_t) \geq p$ for

$i \neq t \neq j$, then $[a] = 1$, and if $\delta(c_i) > p$ or $\delta(c_j) > p$ or $\delta(c_i) + \delta(c_j) = 2p$, then $[a] = 1$ by 1.3.3(iii).

So we assume that $\delta(c_t) < p$ for $i \neq t \neq j$, and $\delta(c_i) \leq p$, $\delta(c_j) \leq p$ and $\delta(c_i) + \delta(c_j) < 2p$. If $\delta(c_j) < p$, then $[a] = [a_1]$ where $a_1 = (c_i, c_j, \delta) \in B(\underline{\underline{b}}_r(p);p)$. If $\delta(c_j) = p$ and $c_i = \max \operatorname{supp} \delta$, then again $[a] = [a_1]$ where $a_1 \in B(\underline{\underline{b}}_r(p);p)$. If $\delta(c_j) = p$ and $c_s = \max \operatorname{supp} \delta \neq c_i$, then by 1.3.3.(iv) $[a] = [[c_i],[c_j],\delta] = [[c_s],[c_j],\delta] = [a_2]$ where $a_2 \in B(\underline{\underline{b}}_r(p);p)$. This completes the proof of the theorem.

We now have a generating set for $\underline{\underline{AA}}_p(G_r(p))$ and we will enlarge it, first to find a generating set for $\gamma_3(G_r(p))$, and then finally to find a generating set for $\underline{\underline{T}}_p(G_r(p))$.

4.4.9 <u>Lemma</u>: Let $w \in \gamma_3(G_r(p))$. Then $w = b_1^{e_1} \ldots b_s^{e_s} t$ where $t \in \underline{\underline{AA}}_p(G_r(p))$ and $b_i \in \{B(\underline{\underline{g}}_r(p);p) \backslash \underline{\underline{a}}(\underline{\underline{g}}_r(p))\}\phi(p)$, $i = 1, \ldots, s$.

<u>Proof</u>: By 4.4.2 $w = b_1^{e_1} \ldots b_s^{e_s} t$ where $t \in \underline{\underline{AA}}_p(G_r(p))$ and $b_i \in B(\underline{\underline{g}}_r(p);p)\phi(p)$. Suppose $b_1, \ldots, b_k \in \underline{\underline{a}}(\underline{\underline{g}}_r(p))\phi(p)$, $b_{k+1}, \ldots, b_s \in \{B(\underline{\underline{g}}_r(p);p) \backslash \underline{\underline{a}}(\underline{\underline{g}}_r(p))\}\phi(p)$. Then we can write

$$w = \prod_{i>j} [g_i, g_j]^{\alpha_{ij}} b_{k+1}^{e_{k+1}} \ldots b_s^{e_s} t \quad \text{where } 0 \leq \alpha_{ij} < p^2.$$

But then $w\gamma_3(G_r(p)) = \prod_{i>j} [g_i, g_j]^{\alpha_{ij}} \gamma_3(G_r(p)) \neq \gamma_3(G_r(p))$ and $w \notin \gamma_3(G_r(p))$ contrary to assumption.

4.4.10 <u>Lemma</u>: Let $u \in \underline{\underline{T}}_p(G_r(p))$ for $p \neq 2$. Then

$$u = \prod_{i=1}^{r} (g^p)^{\alpha_i} \prod_{i>j} ([g_i, g_j]^p) w \quad \text{where } 0 \leq \alpha_i < p, \ 0 \leq \beta_{ij} < p \text{ and}$$

$w \in \gamma_3(G)$.

If $u \in \underline{\underline{T}}_2(G_r(2))$, then $u = \prod_{i=1}^{r} (g_i^4)^{\alpha_i} \prod_{i>j} ([g_i, g_j]^2)^{\beta_{ij}} w$, where $0 \leq \alpha_i < 2$, $0 \leq \beta_{ij} < 2$, and $w \in \gamma_3(G_r(2))$.

<u>Proof</u>: We consider first the case $p \neq 2$. Then $u$ can be written $u = u_1 v$ where $u_1 \in \underline{\underline{B}}_p(G_r(p))$ and $v \in \gamma_3(G_r(p))$. Now $u_1 = u_{11}^p \ldots u_{1\ell}^p$ where $u_{1i} = g_1^{\delta_1} \ldots g_r^{\delta_r} \prod_{i>j} [g_i,g_j]^{\gamma_{ij}} x$ where $x \in \gamma_3(G_r(p))$, $0 \leq \delta_i < p^2$ and $0 \leq \gamma_{ij} < p^2$.

$$\text{Then } u_{1i}^p = \{g_1^{\delta_1} \ldots g_r^{\delta_r} \prod_{i>j} [g_i,g_j]^{\gamma_{ij}} x\}^p$$

$$= \{g_1^{\delta_1} \ldots g_r^{\delta_r}\}^p \{\prod_{i>j} [g_i,g_j]^{\gamma_{ij}}\}^p \bmod \gamma_3(G).$$

Now $(g_1^{\delta_1} \ldots g_r^{\delta_r})^p = (g_1^{\delta_1})^p \ldots (g_r^{\delta_r})^p \prod_{i>j} ([g_i , g_j]^p)^{k_{ij}} \bmod \gamma_3(G)$

since $G/\gamma_3(G)$ is regular, and

$$(\prod_{i>j} [g_i,g_j]^{\gamma_{ij}})^p = \prod_{i>j} ([g_i,g_j]^{\gamma_{ij}})^p \bmod \gamma_3(G).$$

Thus $u_{1i}^p = (g_1^p)^{\delta_1} \ldots (g_r^p)^{\delta_r} \prod_{i>j} ([g_i,g_j]^p)^{\gamma_{ij}+k_{ij}} \bmod \gamma_3(G)$ for $i = 1,\ldots,\ell$, and

$$u_1 = u_{11}^p \ldots u_{1\ell}^p = (g_1^p)^{\alpha_1} \ldots (g_r^p)^{\alpha_r} \prod_{i>j} ([g_i,g_j]^p)^{\beta_{ij}} \bmod \gamma_3(G),$$

which gives the required result for $p \neq 2$.

If $u \in \underline{\underline{T}}_2(G_r(2))$, $u$ can be written $u = vw_1$ where $v \in \underline{\underline{B}}_4(G_r(2))$ and $w_1 \in \gamma_3(G_r(2))$. Now $v = v_1^4 \ldots v_\ell^4$ where $v_i = g_1^{\delta_1} \ldots g_r^{\delta_r} t$ and $t \in \gamma_2(G)$, $0 \leq \delta_i < 8$. Then

$$v_i^4 = (g_1^{\delta_1} \ldots g_r^{\delta_r} t)^4$$

$$= (g_1^{\delta_1} \ldots g_r^{\delta_r})^4 t^4 \bmod \gamma_3(G_r(2))$$

$$= (g_1^{\delta_1} \ldots g_r^{\delta_r})^4 \bmod \gamma_3(G_r(2)) \text{ since } t^4 = 1.$$

If $r = 2$ we have $(g_1^{\delta_1} g_2^{\delta_2})^4 = (g_1^{\delta_1})^4 (g_2^{\delta_2})^4 \ [g_2,g_1]^{2\delta_1\delta_2} \bmod \gamma_3(G_r(2))$, and for $r > 2$ an inductive argument gives the result

$$(g_1^{\delta_1} \ldots g_r^{\delta_r})^4 = (g_1^4)^{\delta_1} \ldots (g_r^4)^{\delta_r} \prod_{i>j} ([g_i,g_j]^2)^{\gamma_{ij}} \bmod \gamma_3(G_r(2)).$$

Hence $v = v_1^4 \cdots v_\ell^4 = \prod_{i=1}^{r} (g_i^4)^{\alpha_i} \prod_{i>j} ([g_i, g_j]^2)^{\beta_{ij}} \bmod \gamma_3(G_r(2))$, which is the required result.

The proof of 4.2.3 now follows directly from 4.4.8, 4.4.9 and 4.4.10.

# Chapter 5

## The Subvarieties of $\mathbf{A}_p\mathbf{T}_p \wedge \mathbf{T}_p\mathbf{A}_p$ .

In this chapter we turn our attention to the variety $\mathbf{A}_p\mathbf{T}_p \wedge \mathbf{T}_p\mathbf{A}_p$ and we find some information about the subvarieties of $\mathbf{A}_p\mathbf{T}_p \wedge \mathbf{T}_p\mathbf{A}_p$. The main results are stated in section 5.1. In section 5.2 we determine the relationship between the free group of $\mathbf{A}_p\mathbf{T}_p \wedge \mathbf{T}_p\mathbf{A}_p$ and the free group of $\mathbf{A}_p\mathbf{T}_p$ and in 5.3 we give a proof of the results in 5.1 modulo several principal lemmas, which are proved in the remaining sections.

## 5.1 Statement of the Main Results

The main results are stated in the form of two theorems, one concerning the subvarieties of $\mathbf{A}_p\mathbf{T}_p \wedge \mathbf{T}_p\mathbf{A}_p$ where $p$ is an odd prime, and the other concerning the subvarieties of $\mathbf{A}_2\mathbf{T}_2 \wedge \mathbf{T}_2\mathbf{A}_2$.

5.1.1 Theorem: Let $p$ be any odd prime, and let $\mathbf{V}$ be a proper subvariety of $\mathbf{A}_p\mathbf{T}_p \wedge \mathbf{T}_p\mathbf{A}_p$. If $\mathbf{V} \geq \mathbf{A}_p\mathbf{A}_p$, then $\mathbf{V} \leq [\mathbf{A}_p\mathbf{A}_p, k\mathbf{E}]$ for some $k \in I$. Otherwise $\mathbf{V}$ is nilpotent. Further $\mathbf{A}_p\mathbf{T}_p \wedge \mathbf{T}_p\mathbf{A}_p \not\leq [\mathbf{A}_p\mathbf{A}_p, k\mathbf{E}]$ for any $k \in I$.

5.1.2 Theorem: Let $\mathbf{V}$ be a proper subvariety of $\mathbf{A}_2\mathbf{T}_2 \wedge \mathbf{T}_2\mathbf{A}_2$.

(i) If $\mathbf{V} \not\geq \mathbf{A}_2\mathbf{A}_2$, then $\mathbf{V}$ is nilpotent.

(ii) If $\mathbf{V} \geq \mathbf{A}_2\mathbf{A}_2$ and $\mathbf{V} \not\geq \mathbf{A}_2\mathbf{T}_2 \wedge \mathbf{T}_2\mathbf{A}_2 \wedge \mathbf{A}_2\mathbf{A}$, then $\mathbf{V} \leq [\mathbf{A}_2\mathbf{A}_2, k\mathbf{E}]$ for some $k \in I$.

(iii) If $\mathbf{V} \geq \mathbf{A}_2\mathbf{T}_2 \wedge \mathbf{T}_2\mathbf{A}_2 \wedge \mathbf{A}_2\mathbf{A}$, then $\mathbf{V} \leq [\mathbf{A}_2\mathbf{A}, h\mathbf{E}]$ for some $h \in I$.

Also, $\mathbf{A}_2\mathbf{T}_2 \wedge \mathbf{T}_2\mathbf{A}_2 \not\leq [\mathbf{A}_2\mathbf{A}, k\mathbf{E}]$ for any $k \in I$, and $\mathbf{A}_2\mathbf{T}_2 \wedge \mathbf{T}_2\mathbf{A}_2 \wedge \mathbf{A}_2\mathbf{A} \not\leq [\mathbf{A}_2\mathbf{A}_2, h\mathbf{E}]$ for any $h \in I$.

5.2 Free Groups in $\underline{\underline{A}}_p\underline{\underline{T}}_p \wedge \underline{\underline{T}}_p\underline{\underline{A}}_p$

To prove the results stated in section 5.1 we will be working in verbal subgroups of the free group of $\underline{\underline{A}}_p\underline{\underline{T}}_p \wedge \underline{\underline{T}}_p\underline{\underline{A}}_p$. In this section we look at the relationship between this free group and the free group of $\underline{\underline{A}}_p\underline{\underline{T}}_p$, and, using the basis theorem from Chapter 4, we determine a basis for $\underline{\underline{T}}_p(F_\infty(\underline{\underline{A}}_p\underline{\underline{T}}_p \wedge \underline{\underline{T}}_p\underline{\underline{A}}_p))$.

Throughout this chapter we will use the following notation:

Let $H(p) = F_\infty(\underline{\underline{A}}_p\underline{\underline{T}}_p \wedge \underline{\underline{T}}_p\underline{\underline{A}}_p)$ and let $\underline{\underline{h}}(p) = \{h_{pi} : i \in I^+\}$ be a free generating set for $H(p)$ which we assume is well-ordered by its indexing set.

To prove the result stated in 5.1.2 we need to work in another free group, namely, $K = F_\infty(\underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}})$, where $\underline{\underline{k}} = \{k_i : i \in I^+\}$ is a free generating set for $K$ which we again assume is well-ordered by its indexing set.

For $p \neq 2$ put $\underline{\underline{b}}_H(p) = \underline{\underline{a}}(\underline{\underline{h}}(2)) \cup (\underline{\underline{h}}(2))^2$, $b_K = \underline{\underline{a}}(\underline{\underline{k}}) \cup (\underline{\underline{k}})^2$. We assume that $\underline{\underline{b}}_H(2)$ and $b_K$ are ordered in a similar way to $\underline{\underline{b}}(2)$ in section 4.1.

Then we have the following results:

5.2.1 Lemma: Let $\phi(p)$ be the valuation mapping $\phi(p) : R(p) \to G(p)$. Then

$$\underline{\underline{N}}_2\underline{\underline{A}}_p(G(p)) = gp(\{(u,c_1,\ldots,c_k) : u \in B(\underline{\underline{g}}(p);p), c_i \in \underline{\underline{b}}(p), k \in I^+\} \cap R(p))\phi(p).$$

Proof: Noting that $\underline{\underline{N}}_2\underline{\underline{A}}_p(G(p)) = \bigcup_{r=1}^{\infty} \underline{\underline{N}}_2\underline{\underline{A}}_p(G_r(p))$, the proof follows directly from 4.4.7 and 4.4.8.

5.2.2 Lemma: For $p \neq 2$ let $V(p) = gp(\underline{\underline{a}}(\underline{\underline{g}}(p))^p\phi(p).\underline{\underline{N}}_2\underline{\underline{A}}_p(G(p))$, and let $V(2) = \underline{\underline{N}}_2\underline{\underline{A}}_2(G(2))$. Then $V(p) = \underline{\underline{T}}_p\underline{\underline{A}}_p(G(p))$ and $G(p)/V(p) \cong H(p)$.

Proof: We prove the result first for $p \neq 2$. We note that $\underline{\underline{T}}_p\underline{\underline{A}}_p(G(p)) = \underline{\underline{B}}_p\underline{\underline{A}}_p(G(p)).\underline{\underline{N}}_2\underline{\underline{A}}_p(G(p))$, so that $V(p) \leq \underline{\underline{T}}_p\underline{\underline{A}}_p(G(p))$.

To prove the reverse inclusion, let $u \in \underline{\underline{T}}_p\underline{\underline{A}}_p(G(p))$. Then $u = vw$, where $v \in \underline{\underline{B}}_p\underline{\underline{A}}_p(G(p))$ and $w \in \underline{\underline{N}}_2\underline{\underline{A}}_p(G(p))$. Since $v \in \underline{\underline{B}}_p\underline{\underline{A}}_p(G(p))$, $v = v_1^p \ldots v_s^p$, where $v_i \in \underline{\underline{A}}_p(G(p))$, $i \in \{1,\ldots,s\}$. Therefore

$$v_i = (g_1^p)^{\alpha_{i1}} \ldots (g_r^p)^{\alpha_{ir}} b_{i1}^{e_{i1}} \ldots b_{it}^{e_{it}} x,$$

where $b_{ij} \in B(\underline{\underline{g}}(p);p)\phi(p)$, $x \in \underline{\underline{A}}\underline{\underline{A}}_p(G(p))$, and $r$ is an integer sufficiently large so that $v_i \in G_r(p)$, for some integers $\alpha_{i1},\ldots,\alpha_{ir}$, $e_{is},\ldots,e_{it}$.

Then

$$\begin{aligned}(v_i)^p &= ((g_1^p)^{\alpha_{i1}} \ldots (g_r^p)^{\alpha_{ir}} b_{i1}^{e_{i1}} \ldots b_{it}^{e_{it}} x)^p \\ &= (g_1^p)^{\alpha_{i1}p} \ldots (g_r^p)^{\alpha_{ir}p} (b_{i1}^{e_{i1}})^p \ldots (b_{it}^{e_{is}})^p t^p \text{ modulo } \underline{\underline{N}}_2\underline{\underline{A}}_p(G(p)).\end{aligned}$$

But $(g_i^p)^{\alpha_{ij}p} = 1$, $t^p = 1$, and $(b_{ij}^{e_{ij}})^p = 1$ unless $b_{ij} \in (\underline{\underline{a}}(\underline{\underline{g}}(p)))\phi(p)$. So $v_i^p \in gp\{(\underline{\underline{a}}(\underline{\underline{g}}(p))^p\phi(p)\}.\underline{\underline{N}}_2\underline{\underline{A}}_p(G(p))$, for $i = 1,\ldots,s$ and hence $v \in V(p)$.

For $p = 2$, $\underline{\underline{T}}_2\underline{\underline{A}}_2(G(2)) = \underline{\underline{N}}_2\underline{\underline{A}}_2(G(2)).\underline{\underline{B}}_4\underline{\underline{A}}_2(G(2))$. Thus $V(2) \leq \underline{\underline{T}}_2\underline{\underline{A}}_2(G(2))$, and to prove the reverse inclusion we have only to show that $\underline{\underline{B}}_4\underline{\underline{A}}_2(G(2)) \leq \underline{\underline{N}}_2\underline{\underline{A}}_2(G(2))$. Let $v \in \underline{\underline{B}}_4\underline{\underline{A}}_2(G(2))$. Then Then $v = v_1^4 \ldots v_s^4$ where $v_i \in \underline{\underline{A}}_2(G(2)) = \underline{\underline{B}}_2(G(2))$, $i = 1,\ldots,s$. But $v_i = (v_{i1}^2 \ldots v_{it}^2)$ for some $v_{ij} \in G(2)$, $j = 1,\ldots,t$, so that $v_i^4 = (v_{i1}^2 \ldots v_{it}^2)^4 = (v_{i1}^2)^4 \ldots (v_{it}^2)^4 \bmod \underline{\underline{N}}_2\underline{\underline{A}}_2(G(2)) = 1 \bmod \underline{\underline{N}}_2\underline{\underline{A}}_2(G(2))$. Hence $v_i^4 \in \underline{\underline{N}}_2\underline{\underline{A}}_2(G(2))$ and $v \in V(2)$.

In the next theorem we will use the following notation: For $p \neq 2$ let $Q_H(p) = \{(b,c) : b \in (B(\underline{\underline{h}}(p);p)\backslash\underline{\underline{a}}(\underline{\underline{h}}(p)))^p, c \in \underline{\underline{a}}(\underline{\underline{h}}(p))\}$, and $R_H(p) = (\underline{\underline{h}}(p))^p \cup (B(\underline{\underline{h}}(p));p)\backslash\underline{\underline{a}}(\underline{\underline{h}}(p))) \cup \underline{\underline{a}}(\underline{\underline{b}}_H(p)) \cup Q_H(p)$, and let $Q_H(2) = \{(b,c) : b \in B(\underline{\underline{h}}(2);2)\backslash\underline{\underline{a}}(\underline{\underline{h}}(2)), c \in \underline{\underline{a}}(\underline{\underline{h}}(2))\}$, and

$R_H(2) = (\underline{\underline{h}}(2))^4 \cup (\underline{\underline{a}}(\underline{\underline{h}}(2)))^2 \cup (B(\underline{\underline{h}}(2);2)\backslash \underline{\underline{a}}(\underline{\underline{h}}(2)) \cup \underline{\underline{a}}(\underline{\underline{b}}_H(2)) \cup Q_H(2)$.

Then we have the following result:

5.2.3 Theorem: The valuation mapping $\phi_H(p) : R_H(p) \to H(p)$ is one-to-one and $R_H(p)\phi_H(p)$ is a basis for $\underline{\underline{T}}_p(H(p))$.

Proof: The proof follows immediately by noting that any contradiction to 5.2.3 would provide a contradiction to 4.1.2 since $H(p) \cong G(p)/V(p)$.

As we noted above, for the case $p = 2$ we need to work in the free group of $\underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}}$ and in the following lemma we look at this as a quotient group of $H(2)$.

5.2.4 Lemma: In $C(H(2))$ let

$\underline{\underline{w}} = (\underline{\underline{a}}(\underline{\underline{h}}(2)))^2 \cup \{(b,c);b \in B(\underline{\underline{h}}(2);2), c \in \underline{\underline{a}}(\underline{\underline{h}}(2))\}$, and let $W = \mathrm{gp}\ \underline{\underline{w}}\phi_H(2)$. Then $W = \underline{\underline{A}}_2\underline{\underline{A}}(H(2))$ and $H(2)/W \cong K$.

Proof: $\underline{\underline{A}}_2\underline{\underline{A}}(H(2)) = \underline{\underline{AA}}(H(2)).\underline{\underline{B}}_2\underline{\underline{A}}(H(2))$, so that $\underline{\underline{A}}_2\underline{\underline{A}}(H(2)) \geq W$.

To prove the reverse inclusion we show that $W \geq \underline{\underline{AA}}(H(2))$ and $W \geq \underline{\underline{B}}_2\underline{\underline{A}}(H(2))$. If $v \in \underline{\underline{A}}(H(2))$, $v = b_1^{e_1}\dots b_s^{e_s}t$, where $b_i \in B(h(2);2)$, $t \in \underline{\underline{AA}}_2(H(2))$ and $e_i \in \{1,2,3\}$, $i = 1,\dots,s$, so that $\underline{\underline{AA}}(H(2))$ is generated by elements $[v,u]$ where $v,u \in \underline{\underline{A}}(H(2))$. Let $u = d_1^{c_1}\dots d_{s'}^{c_{s'}}r$ where $d_i \in B(\underline{\underline{h}}(2);2)$, $r \in \underline{\underline{AA}}_2(H(2))$, $c_i \in \{1,2,3\}$, $i = 1,\dots,s'$.

Then $[v,u] = [b_1^{e_1}\dots b_s^{e_s}t, d_1^{c_1}\dots d_{s'}^{c_{s'}}r]$

$$= \prod_{\substack{1\leq i\leq s\\ 1\leq j\leq s'}} [b_i,d_j]^{e_ic_j}.$$

But if both $b_i,d_j \in \underline{\underline{N}}_2(H(2))$, $[b_i,d_j] = 1$, so that the only non-trivial terms in this product are those for which at least one of $b_i$ and $d_j$ are in $(\underline{\underline{a}}(\underline{\underline{h}}(2)))\phi_H(2)$ and in this case $[b_i,d_j] \in W$ and hence

$[v,u] \in W$ and $\underline{\underline{AA}}(H(2)) \leq W$.

We now show that $\underline{\underline{B}}_2\underline{\underline{A}}(H(2)) \leq W$. $\underline{\underline{B}}_2\underline{\underline{A}}(H(2))$ is generated by elements $v^2$, where $v \in \underline{\underline{A}}(H(2))$. Then

$$
\begin{aligned}
v^2 &= (b_1^{e_1} \ldots b_s^{e_s} t)^2 \\
&= (b_1^{e_1} \ldots b_s^{e_s})^2 \\
&= (b_1^{e_1})^2 \ldots (b_s^{e_s})^2 \quad \text{modulo } \underline{\underline{AA}}(H(2)) \\
&= (b_1^2)^{e_1} \ldots (b_s^2)^{e_s} \quad \text{modulo } \underline{\underline{AA}}(H(2)).
\end{aligned}
$$

But only these $b_i^2$ are non-trivial for which $b_i \in (\underline{\underline{a}}(\underline{\underline{h}}(2)))\phi_H(2)$. Thus, since $\underline{\underline{AA}}(H(2)) \leq W$, $v^2 \in W$ and $\underline{\underline{B}}_2\underline{\underline{A}}(H(2)) \leq W$.

5.2.6 <u>Lemma</u>: In $C(K)$ let

$$R_K = (\underline{\underline{k}})^4 \cup (B(\underline{\underline{k}};2)\setminus \underline{\underline{a}}(\underline{\underline{k}})) \cup \underline{\underline{a}}((\underline{\underline{k}})^2) \cup \{(b,c) : b \in B(\underline{\underline{k}};2), c \in (\underline{\underline{k}})^2\}.$$

Then the valuation mapping $\phi_K : R_K \to K$ is one-to-one and $R_K\phi_K$ is a basis for $\underline{\underline{T}}_p(K)$.

<u>Proof</u>: Since $K \cong H(2)/W$ any contradiction to this result would also provide a contradiction to 5.2.3.

It will be necessary to know what the $\underline{\underline{A}}_p\underline{\underline{A}}_p$ subgroup of these groups consists of.

5.2.7 <u>Lemma</u>: For $p \neq 2$, let $\underline{\underline{u}}(p) = Q_H(p) \cup \underline{\underline{a}}(\underline{\underline{b}}_H(p))$ and let $U(p) = \text{gp } \underline{\underline{u}}(p)\phi_H(p)$. Then $U(p) = \underline{\underline{A}}_p\underline{\underline{A}}_p(H(p))$.

<u>Proof</u>: $\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)) = \underline{\underline{AA}}_p(H(p)).\underline{\underline{B}}_p\underline{\underline{A}}_p(H(p))$ so that $\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)) \geq U(p)$. Recalling that $H(p) \cong G(p)/V(p)$ we have immediately that $\underline{\underline{B}}_p\underline{\underline{A}}_p(H(p)) = \{1\}$. However, using the relationship between $H(p)$ and $G(p)$ again, and using 4.4.8, we have that $U(p) = \underline{\underline{AA}}_p(H(p))$, giving the required result.

5.2.8 Lemma: In $C(K)$, let
$\underline{\underline{u}}(2) = (\underline{\underline{k}})^4 \cup \underline{\underline{a}}((k)^2) \cup \{(b,c) : b \in B(\underline{\underline{k}};2),\ c \in (\underline{\underline{k}})^2\}$, and let
$U(2) = \mathrm{gp}\ \underline{\underline{u}}(2)\phi_K$. Then $U(2) = \underline{\underline{A}}_2\underline{\underline{A}}_2(K)$.

Proof: $\underline{\underline{A}}_2\underline{\underline{A}}_2(K) = \underline{\underline{AA}}_2(K).\underline{\underline{B}}_2\underline{\underline{A}}_2(K)$, so that $\underline{\underline{A}}_2\underline{\underline{A}}_2(K) \supset U(2)$. To prove the reverse inclusion we have only to show that $U(2) \geq \underline{\underline{AA}}_2(K)$ and $U(2) \geq \underline{\underline{B}}_2\underline{\underline{A}}_2(K)$. By considering the relationship between $G(2)$ and $K$, and using 4.4.8, we have that $\underline{\underline{AA}}_2(K)$ is generated by $\underline{\underline{a}}((\underline{\underline{k}})^2) \cup \{(b,c) : b \in B(\underline{\underline{k}};2),\ c \in (k)^2\}\phi_K$, so that $U(2) \geq \underline{\underline{AA}}_2(K)$.

To show that $U(2) \geq \underline{\underline{B}}_2\underline{\underline{A}}_2(K)$ we note that $\underline{\underline{B}}_2\underline{\underline{A}}_2(K)$ is generated by elements $u^2$ where $u \in \underline{\underline{A}}_2(K)$. We can write $u = (k_1^2)^{\alpha_1}\ldots(k_r^2)^{\alpha_r} b_1^{e_1}\ldots b_s^{e_s}\, t$ where $b_i \in B(k;2)\phi_K$, $t \in \underline{\underline{AA}}_2(K)$, $0 \leq \alpha_i \leq 3$, $1 \leq e_j \leq 3$. Then

$$u^2 = ((k_1^2)^{\alpha_1}\ldots(k_r^2)^{\alpha_r} b_1^{e_1}\ldots b_s^{e_s}\, t)^2$$

$$= (k_1^4)^{\alpha_1}\ldots(k_r^4)^{\alpha_r}(b_1^2)^{e_1}\ldots(b_s^2)^{e_s}\, t^2 \quad \text{modulo } \underline{\underline{AA}}_2(K).$$

But $b_i^2 = 1$, $i \in \{1,\ldots,s\}$, and $t^2 = 1$ so that $u^2 = (k_1^4)^{\alpha_1}\ldots(k_r^4)^{\alpha_r}$ modulo $\underline{\underline{AA}}_2(K)$. Therefore $u^2 \in U(2)$ and $\underline{\underline{B}}_2\underline{\underline{A}}_2(K) \leq U(2)$ which gives the required result.

The following relationships between some of the verbal subgroups will prove useful.

5.2.9 Lemma: For $p \neq 2$, $[\underline{\underline{AA}}(H(p)),kH(p)] \geq [\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)),(k+1)H(p)]$.

Proof: The proof is by induction on $k$. In the first step we prove the result for $k = 0$. Thus we have to show that $[\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)),H(p)] \leq \underline{\underline{AA}}(H(p))$. By 5.2.6 $[\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)),H(p)]$ is generated by elements $[[b,c],h]$ where $b \in \underline{\underline{A}}(H(p))$, or $b = h_{pi}^p$ for some

$i \in I$, $c \in \underline{\underline{A}}(H(p))$ and $h \in H(p)$. If $b \in \underline{\underline{A}}(H(p))$, then $[[b,c],h] \in \underline{\underline{AA}}(H(p))$. Otherwise we have $[[b,c],h] = [[h^p_{pi},c],h] = [[h^p_{pi},h],c]$ by 1.2.3(ii) so that $[[b,c],h] \in \underline{\underline{AA}}(H(p))$. The rest of the induction follows immediately.

5.2.10 Lemma: Let $k \in I$, then

$$[\underline{\underline{AA}}(H(2)),kH(2)] \geq [\underline{\underline{A}}_2\underline{\underline{A}}(H(2)), (k+1)H(2)].$$

Proof: Let $k = 0$. Then by 5.2.4 $[\underline{\underline{A}}_2\underline{\underline{A}}(H(2)),H(2)]$ is generated by elements $[[b,c],h]$ or $[b^2,h]$ where $b,c \in \underline{\underline{A}}(H(2))$ and $h \in H(2)$. But $[b,c,h] \in \underline{\underline{AA}}(H(2))$, and $[b^2,h] = [[b,h],b] \in \underline{\underline{AA}}(H(2))$, so that $\underline{\underline{AA}}(H(2)) \geq [\underline{\underline{A}}_2\underline{\underline{A}}(H(2)),H(2)]$. An easy induction completes the proof.

5.2.11 Lemma; Let $r \in I$. Then

$$[\underline{\underline{A}}(K),\underline{\underline{A}}_2(K),rK] \geq [\underline{\underline{A}}_2\underline{\underline{A}}_2(K),(r+1)K].$$

Proof: Let $r = 0$. By 5.2.8 $[\underline{\underline{A}}_2\underline{\underline{A}}_2(K),K]$ is generated by elements of the form $[k_i^4,k],[[k_i^2,k_j^2],k]$ or $[[b,k_i^2],k]$ where $b \in \underline{\underline{A}}(K), k \in K$ and $k_i,k_j \in \underline{\underline{k}}$. But $[k_i^4,k] = [[k_i^2,k],k_i^2]$ which is in $[\underline{\underline{A}}(K),\underline{\underline{A}}_2(K)]$, $[[k_i^2,k_j^2],k] = [[k_i^2,k],k_j^2][[k_j^2,k]k_i^2] \in [\underline{\underline{A}}(K),\underline{\underline{A}}_2(K)]$, and $[[b,k_i^2],k] \in [\underline{\underline{A}}(K),\underline{\underline{A}}_2(K)]$. Thus $[\underline{\underline{A}}(K),\underline{\underline{A}}_2(K)] \geq [\underline{\underline{A}}_2\underline{\underline{A}}_2(K),K]$ and the result for $r \geq 1$ follows by induction.

## 5.3 Skeletal Proofs of 5.1.1 and 5.1.2

This section comprises a series of lemmas which culminate in the proofs of 5.1.1 and 5.1.2. For the sake of simplicity, presentation of the proofs of four of the lemmas is postponed until later sections, but apart from these the argument is complete.

We will be working in the $\underline{\underline{T}}_p$ subgroups of $H(p)$ for all primes $p$, and $K$. For convenience of notation we will let $G$ represent $H(p)$ or $K$ in the following definitions. Theorem 4.1.2 and the results of 5.2 tell us that $\underline{\underline{T}}_p(G)$ is free abelian of exponent $p$, and the basis that has been exhibited for $\underline{\underline{T}}_p(G)$ enables us to express elements of $\underline{\underline{T}}_p(G)$ in canonic form. We have already given a definition of "normal" form for elements of a free abelian subgroup, but for the sake of clarity we give the definition for the specific cases we are working with.

5.3.1 Definition: Let $w \in \underline{\underline{T}}_p(G)$. Then $w$ is expressed in normal form when written $w = b_1^{e_1} \dots b_s^{e_s}$ where $b_1, \dots, b_s$ are distinct members of the basis for $\underline{\underline{T}}_p(G)$ and $e_1, \dots, e_s$ are integers satisfying $e_j \not\equiv 0 \bmod p$ for each $j \in \{1, \dots, s\}$.

It is obvious that an expression of an element of $\underline{\underline{T}}_p(G)$ in normal form is unique up to the arrangement of the product and congruence modulo $p$ of the integers.

We want to distinguish some elements of $\underline{\underline{T}}_p(G)$ as "special" and we do this in the following definition. The terminology is that of Brooks [8] and "special" elements here serve a similar purpose to his "special" elements, but the definition here is different because of the more complex situation. In the following definitions we let $\underline{\underline{g}} = \{g_i : i \in I^+\}$ represent the free generating set for $H(p)$ or $K$.

5.3.2 Definition: Let $t = (b,c)$ be a commutator in $C(G)$ where $b = (u,v,\delta) \in B(\underline{\underline{g}};p)$ and either $c \in \underline{\underline{a}}(\underline{\underline{g}})$, or $c \in (\underline{\underline{g}})^2$. Then $t$ is called special if, and only if

1) $\operatorname{supp} \delta \subseteq \underline{\underline{g}}$

2) $(u,v,\delta) = (g_2, g_1, g_3, \dots)$

3) $\delta(g_1) = \delta(g_2) = \delta(g_3) = 1$

4) (i) if $c \in \underline{\underline{a}}(\underline{\underline{g}})$, $c = (g_i, g_j)$, then $i, j > 3$,

(ii) if $c \in (\underline{\underline{g}})^2$, $c = (g_k)^2$, then $k > 3$.

A commutator-element $x \in G$ is called special if $x = [t]$ for a special commutator $t$. If $w \in \underline{\underline{T}}_p(G)$ is expressed in normal form by $b_1^{e_1} \ldots b_s^{e_s}$ then $w$ is special if $b_i$ is special, for $i \in \{1, \ldots, s\}$.

Since for certain considerations special elements are particularly convenient, it is useful to have a method of obtaining special elements from arbitrary ones.

5.3.3 Definition: Let $\tau : G \to G$ and $\kappa_i : G \to G$ for $i \in I^+$, be the endomorphisms of $G$ induced respectively by the maps

$\bar{\tau} : \underline{\underline{g}} \to \underline{\underline{g}}$; $g\,\bar{\tau} = g_{j+3}$ for all $j \in I^+$

$$\bar{\kappa}_i : \underline{\underline{g}} \to G; \quad g_j \bar{\kappa}_i = \begin{cases} g_j [g_2, g_1, g_3] & \text{if } j = i \\ g_j & \text{otherwise.} \end{cases}$$

Then for $w \in \underline{\underline{T}}_p(G)$, and all $i \in I^+$, define $w^{(i)}$ by

$$w^{(i)} = (w\tau\kappa_{i+3})(w\tau)^{-1}.$$

We now give the statement of one of the principal lemmas of this section. The proof is postponed until 5.4.

5.3.4 Lemma: (i) For $p \neq 2$, all $w \in \underline{\underline{A}}_p\underline{\underline{A}}_p(H(p))$, and all $i \in I^+$, $w^{(i)}$ is special, and if $w$ is non-trivial so is $w^{(i)}$ for at least one value of $i$.

(ii) For all $w \in \underline{\underline{A}}_2\underline{\underline{A}}(H(2))$, and all $i \in I^+$, $w^{(i)}$ is special, and if $w$ is non trivial so is $w^{(i)}$ for at least one value of $i$.

(iii) For all $w \in \underline{\underline{A}}_2\underline{\underline{A}}_2(K)$, and all $i \in I^+$, $w^{(i)}$ is special, and if $w$ is non-trivial so is $w^{(i)}$ for at least one value of $i$.

This completes the preparatory remarks about elements of $G$, where as before $G$ denotes $H(p)$ or $K$. The information about $G$ required to prove the results in 5.1 concerns the verbal subgroups of $G$, and in this connection the following notation will be used: the lattice of fully invariant subgroups of $G$ will be denoted by $\mathrm{lat}(G)$, and if $U \in \mathrm{lat}(G)$ then $\mathrm{id}(U)$ denotes the ideal in $\mathrm{lat}(G)$ generated by $U$; i.e. $\mathrm{id}\, U = \{V \in \mathrm{lat}(G) \mid V \leq U\}$. Also, for convenience in writing we put $\mathrm{id}^{\#}(U) = \mathrm{id}(U)\backslash\{\{1\}\}$.

The lattice dual-isomorphism $\mu_G : \mathrm{lat}(\mathrm{var}G) \to \mathrm{lat}(G)$ defined by $\underline{\underline{V}}\mu_G = \underline{\underline{V}}(G)$ for all $\underline{\underline{V}} \in \mathrm{lat}(\mathrm{var}G)$, or more particularly, its inverse, will be employed to interpret statements about $\mathrm{lat}(G)$ as statements about $\mathrm{lat}(\mathrm{var}G)$, and those properties of $\mu_G$ which are described in, or follow immediately from, sections 3 and 4 of Chapter 1 of [17] will often be used without explicit mention.

A major step in the proof of 5.1.1 is the following result. The proof is postponed until section 5.5.

5.3.5 <u>Lemma</u>: Let $w$ be a non-trivial special element of $\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p))$ for $p \neq 2$. Then there exists an integer $e \in I$ such that

$$\langle w\rangle \geq [\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)), e\, H(p)].$$

The above lemma easily generalises to give the following:

5.3.6 <u>Lemma</u>: Let $W \in \mathrm{Id}^{\#}(\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)))$. Then there exists an integer $k \in I$ such that $W \geq [\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)),\ k\, H(p)]$.

<u>Proof</u>: Choose $w \in W$ such that $w \neq 1$. By 5.3.4(i) $w^{(i)}$ is special and non-trivial for some $i \in I^{+}$, and therefore by 5.3.5 we have that

$$W \geq \langle w^{(i)} \rangle \geq [\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)), k\ H(p)]$$

for some $k \in I$.

With these results we can now give a proof of 5.1.1. We note that the part of this theorem that considers a variety $\underline{\underline{V}} \leq \underline{\underline{A}}_p\underline{\underline{T}}_p \wedge \underline{\underline{T}}_p\underline{\underline{A}}_p$ such that $\underline{\underline{V}} \not\geq \underline{\underline{A}}_p\underline{\underline{A}}_p$ could be proved by using the result of Kovács and Newman [14] that the only nilpotent-by-abelian just-non-Cross varieties are $\underline{\underline{A}}$ and $\underline{\underline{A}}_p\underline{\underline{A}}_p$. We prefer not to use this quite strong result, but to use more elementary methods. We do, however, use the result that every proper subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_p$ is nilpotent which is easily seen from the description of the subvarieties of $\underline{\underline{A}}_p\underline{\underline{A}}_p$ given in section 2.4.

<u>Proof of 5.1.1</u>: Let $\underline{\underline{V}}$ be a proper subvariety of $\underline{\underline{A}}_p\underline{\underline{T}}_p \wedge \underline{\underline{T}}_p\underline{\underline{A}}_p$. If $\underline{\underline{V}} \geq \underline{\underline{A}}_p\underline{\underline{A}}_p$, then by 5.3.6 there is an integer $k \in I$ such that $\underline{\underline{V}}\mu_{H(p)} \geq [\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)), k\ H(p)]$ and this gives the required result.

Now consider the case $\underline{\underline{V}} \not\geq \underline{\underline{A}}_p\underline{\underline{A}}_p$. If $\underline{\underline{V}} < \underline{\underline{A}}_p\underline{\underline{A}}_p$ we have immediately that $\underline{\underline{V}}$ is nilpotent. So we assume that $\underline{\underline{V}} \not\leq \underline{\underline{A}}_p\underline{\underline{A}}_p$. Then $\underline{\underline{V}} \wedge \underline{\underline{A}}_p\underline{\underline{A}}_p$ is a proper subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_p$ and, putting $V = \underline{\underline{V}}\mu_{H(p)}$, we can write

5.3.7 $\qquad V.\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)) \geq \gamma_r(H(p))$

for some $r \in I^+$. Also

$$[V.\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)), \underline{\underline{A}}(H(p))] = [V, \underline{\underline{A}}(H(p))][\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)), \underline{\underline{A}}(H(p))],$$

and since $[\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)), \underline{\underline{A}}(H(p))] = 1$, we have

$$V \geq [V.\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)), \underline{\underline{A}}(H(p))] \geq [\gamma_r(H(p)), \underline{\underline{A}}(H(p))].$$

But

$$[\gamma_r(H(p)),\mathbf{A}(H(p))] \geq [\mathbf{A}(H(p)),\mathbf{A}(H(p)),(r-2)H(p)]$$

$$= [\mathbf{A}\mathbf{A}(H(p)),(r-2)H(p)]$$

using several applications of 1.2.2(iii). Thus

$$V \geq [\mathbf{A}\mathbf{A}(H(p)),(r-2)H(p)]$$

$$\geq [\mathbf{A}_p\mathbf{A}_p(H(p)),(r-1)H(p)] \qquad \text{by 5.2.9.}$$

Using 5.3.7 again we have

$$[\gamma_r(H(p)),(r-1)H(p)] \leq [V.\mathbf{A}_p\mathbf{A}_p(H(p)),(r-1)H(p)]$$

$$= [V,(r-1)H(p)][\mathbf{A}_p\mathbf{A}_p(H(p)),(r-1)H(p)]$$

$$\leq V$$

and hence $V \geq \gamma_{2r-1}(H(p))$, or $\mathbf{V}$ is nilpotent as claimed.

It only remains to show that $\mathbf{A}_p\mathbf{T}_p \wedge \mathbf{T}_p\mathbf{A}_p \not\leq [\mathbf{A}_p\mathbf{A}_p, k\,\mathbf{E}]$ for any $k \in I$. But if it were then $[[x,y,z],[u,v],x_1,\dots,x_k]$ would be a law in $H(p)$, and this would imply that the commutator-element

$$[\,h_{p2},h_{p1}\,h_{p3},\dots,h_{p,k+3}],[h_{p,k+4},h_{p,k+5}]]$$

is trivial in $H(p)$ contradicting 5.2.3.

We now consider the subvarieties of $\mathbf{A}_2\mathbf{T}_2 \wedge \mathbf{T}_2\mathbf{A}_2$. The results are similar to those for $\mathbf{A}_p\mathbf{T}_p \wedge \mathbf{T}_p\mathbf{A}_p$ for $p \neq 2$, as would be expected, but they are more complicated, due largely to the fact that $\mathbf{A}_2\mathbf{T}_2 \wedge \mathbf{T}_2\mathbf{A}_2$ has exponent 8, in contrast to $\mathbf{A}_p\mathbf{T}_p \wedge \mathbf{T}_p\mathbf{A}_p$ which has exponent $p^2$.

The following two lemmas give the major steps towards the proof of 5.1.2. The proofs of these lemmas are postponed to section 5.5.

5.3.8 <u>Lemma</u>: Let $w$ be a non-trivial special element of $\underline{\underline{A}}_2\underline{\underline{A}}_2(K)$. Then there exists $e \in I$ such that

$$\langle w\rangle \geq [\underline{\underline{A}}_2\underline{\underline{A}}_2(K), e\ K].$$

5.3.9 <u>Lemma</u>: Let $w$ be a non-trivial special element of $\underline{\underline{A}}_2\underline{\underline{A}}(H(2))$. Then there exists $k \in I$ such that

$$\langle w\rangle \geq [\underline{\underline{A}}_2\underline{\underline{A}}(H(2)), k\ H(2)].$$

These two lemmas are easily generalised in the following way.

5.3.10 <u>Lemma</u>: Let $W \in Id^{\#}(\underline{\underline{A}}_2\underline{\underline{A}}_2(K))$. Then there exists an integer $r \in I$ such that $W \geq [\underline{\underline{A}}_2\underline{\underline{A}}_2(K), r\ K]$.

5.3.11 <u>Lemma</u>: Let $W \in Id^{\#}(\underline{\underline{A}}_2\underline{\underline{A}}(H(2)))$. Then there exists an integer $s \in I$ such that $W \geq [\underline{\underline{A}}_2\underline{\underline{A}}(H(2)), s\ H(2)]$.

The proofs of 5.3.10 and 5.3.11 depend on 5.3.5(ii) and (iii), and are similar to the proof of 5.3.7, so they are not given here.

Before proving 5.1.2 in its entirety we prove the following result concerning the subvarieties of $\underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}}$.

5.3.12 <u>Theorem</u>: Let $\underline{\underline{V}}$ be a proper subvariety of $\underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}}$. If $\underline{\underline{V}} \geq \underline{\underline{A}}_2\underline{\underline{A}}_2$, then there exists $k \in I$ such that $\underline{\underline{V}} \leq [\underline{\underline{A}}_2\underline{\underline{A}}_2, k\ \underline{\underline{E}}]$. Otherwise $\underline{\underline{V}}$ is nilpotent. Further $\underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}} \not\leq [\underline{\underline{A}}_2\underline{\underline{A}}_2, s\ \underline{\underline{E}}]$ for any $s \in I$.

<u>Proof</u>: Let $V = \underline{\underline{V}}\mu_K$. If $\underline{\underline{V}} \geq \underline{\underline{A}}_2\underline{\underline{A}}_2$, then $V \in id^{\#}(\underline{\underline{A}}_2\underline{\underline{A}}_2(K))$ and by 5.3.10 there exists $k \in I$ such that $V \geq [\underline{\underline{A}}_2\underline{\underline{A}}_2(K), k\ K]$, or in other words, $\underline{\underline{V}} \leq [\underline{\underline{A}}_2\underline{\underline{A}}_2, k\ \underline{\underline{E}}]$ as claimed.

Now consider the case $\underline{\underline{V}} \not\geq \underline{\underline{A}}_2\underline{\underline{A}}_2$. If $\underline{\underline{V}} < \underline{\underline{A}}_2\underline{\underline{A}}_2$, then $\underline{\underline{V}}$ is nilpotent. So assume $\underline{\underline{V}} \not\leq \underline{\underline{A}}_2\underline{\underline{A}}_2$. Then $\underline{\underline{V}} \wedge \underline{\underline{A}}_2\underline{\underline{A}}_2$ is a proper subvariety of $\underline{\underline{A}}_2\underline{\underline{A}}_2$ and is nilpotent. Thus we can write

5.3.13 ... $V.\underline{\underline{A}}_2\underline{\underline{A}}_2(K) \geq \gamma_r(K)$ for some $r \in I$.

Therefore,

$$[V.\underline{\underline{A}}_2\underline{\underline{A}}_2(K), \underline{\underline{A}}_2(K)] \geq [\gamma_r(K), \underline{\underline{A}}_2(K)].$$

But

$$[V.\underline{\underline{A}}_2\underline{\underline{A}}_2(K), \underline{\underline{A}}_2(K)] = [V, \underline{\underline{A}}_2(K)][\underline{\underline{A}}_2\underline{\underline{A}}_2(K), \underline{\underline{A}}_2(K)]$$
$$= [V, \underline{\underline{A}}_2(K)] \leq V,$$

and so we have

$$V \geq [\gamma_r(K), \underline{\underline{A}}_2(K)] \geq [\underline{\underline{A}}(K), \underline{\underline{A}}_2(K), (r-2)K],$$

by repeated applications of 1.2.2(iii). But from 5.2.9 this is enough to show that $V \geq [\underline{\underline{A}}_2\underline{\underline{A}}_2(K), (r-1)K]$.

Using 5.3.13 again we have

$$[\gamma_r(K), (r-1)K] \leq [V.\underline{\underline{A}}_2\underline{\underline{A}}_2(K), (r-1)K]$$
$$= [V, (r-1)K][\underline{\underline{A}}_2\underline{\underline{A}}_2(K), (r-1)K] \leq V,$$

and hence $V \geq \gamma_{2r-1}(K)$, or $\underline{\underline{V}}$ is nilpotent as claimed.

To prove the last part of the theorem, suppose that $\underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}} \leq [\underline{\underline{A}}_2\underline{\underline{A}}_2, s\,\underline{\underline{E}}]$ for some $s \in I$. But then $[x^4, y_1, \ldots, y_s]$ would be a law in $K$ which would imply that the commutator-element $[k_1^4, k_2, \ldots, k_{s+1}]$ is trivial.

But

$$[k_1^4, k_2, \ldots, k_{s+1}] = [[k_1^2, k_2, \ldots, k_{s+1}], k_1^2]$$
$$= [[k_2, 2k_1, k_3, \ldots, k_{s+1}], k_1^2],$$

which is a basis element in $\underline{\underline{T}}_2(K)$ by 5.2.6 and so cannot be trivial.

Before proving 5.1.2 there is one final result we need.

5.3.14 <u>Lemma</u>: Let $V$ be a fully invariant subgroup of $H(2)$ such that $V \leq \underline{\underline{A}}_2\underline{\underline{A}}_2(H(2))$. Then $V \wedge \underline{\underline{A}}_2\underline{\underline{A}}(H(2)) \neq \{1\}$.

<u>Proof</u>: If $V \geq \underline{\underline{A}}_2\underline{\underline{A}}(H(2))$ the result is trivial, so assume that $V \not\geq \underline{\underline{A}}_2\underline{\underline{A}}(H(2))$. Put $\underline{\underline{V}} = V\mu^{-1}_{H(2)}$. Then $\underline{\underline{V}} \wedge \underline{\underline{A}}_2\underline{\underline{A}}$ is a proper subvariety of $\underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}}$ and so by 5.3.12 $\underline{\underline{V}} \wedge \underline{\underline{A}}_2\underline{\underline{A}} \leq [\underline{\underline{A}}_2\underline{\underline{A}}_2, k\,E]$ for some $k \in I$, and we can write

$$V.\underline{\underline{A}}_2\underline{\underline{A}}(H(2)) \geq [\underline{\underline{A}}_2\underline{\underline{A}}_2(H(2)), kH(2)].$$

Then there exists $v \in V$, $w \in \underline{\underline{A}}_2\underline{\underline{A}}(H(2))$ such that

$$vw = [[h_2,h_1,h_3,\ldots,h_s],h^2_{s+1}], \quad s \geq k+3,$$

and $h_1,\ldots,h_{s+1} \in \underline{\underline{h}}(2)$.

Then $v = [[h_2,h_1,h_3,\ldots,h_s],h^2_{s+1}]w$,

where $w = w(h_1,\ldots,h_{s+1})$ can be written as a product of basis elements in $\underline{\underline{A}}_2\underline{\underline{A}}(H(2))$, and $v = v(h_1,\ldots,h_{s+1})$. Replace $h_{s+1}$ by $h_{s+1}h_{s+2}$ in this product and expand, so that

$$v(h_1,\ldots,h_s,h_{s+1}h_{s+2}) = v(h_1,\ldots,h_{s+1})v(h_1,\ldots,h_s,h_{s+2})v'(h_1,\ldots,h_{s+2}),$$

$$w(h_1,\ldots,h_s,h_{s+1}h_{s+2}) = w(h_1,\ldots,h_{s+1})w(h_1,\ldots,h_s,h_{s+2})w'(h_1,\ldots,h_{s+2}),$$

and

$$[[h_2,h_1,h_3,\ldots,h_s],(h_{s+1}h_{s+2})^2]$$

$$= [[h_2,h_1,h_3,\ldots,h_s],h^2_{s+1}]\ [[h_2,h_1,h_3,\ldots,h_s],h^2_{s+2}] \times [[h_2,h_1,h_3,\ldots,h_s],[h_{s+2},h_{s+1}]],$$

and this implies that

$$v'(h_1,\ldots,h_{s+2}) = [[h_2,h_1,h_3,\ldots,h_3],[h_{s+2},h_{s+1}]]w'(h_1,\ldots,h_{s+2}).$$

Therefore $v'(h_1,\ldots,h_{s+2}) \in V \wedge \underline{\underline{A}}_2\underline{\underline{A}}(H(2))$, and the proof is complete if we can show that $v'(h_1,\ldots,h_{s+2}) \neq 1$. But this is the case as long as $w'(h_1,\ldots,h_{s+2}) \neq [[h_2,h_1,h_3,\ldots,h_s],[h_{s+2},h_{s+1}]]$.

And this does not happen since $w$ is a product of elements of the form $[h_i,h_j]^2$ and $[b,c]$ where $b = [h_r,h_s,\delta]$ and $c = [h_k,h_\ell]$, and under these circumstances

$$w'(h_1,\dots,h_{s+2}) \neq [[h_2,h_1,h_3,\dots,h_s],[h_{s+2}h_{s+1}]]^{-1}.$$

Now we can finally give a proof of 5.1.2.

<u>Proof of 5.1.2</u>: Let $\underline{\underline{V}}$ be a proper subvariety of $\underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2$. If $\underline{\underline{V}} \geq \underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}}$, then by 5.3.11 $\underline{\underline{V}} \leq [\underline{\underline{A}}_2\underline{\underline{A}}, k\,\underline{\underline{E}}]$ for some $k \in I$.

Assume $\underline{\underline{V}} \not\geq \underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}}$. If $\underline{\underline{V}} \leq \underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}}$, then $\underline{\underline{V}}$ satisfies 5.1.2 by 5.3.12. So assume $\underline{\underline{V}} \not\leq \underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}}$. We consider first the case $\underline{\underline{V}} \not\geq \underline{\underline{A}}_2\underline{\underline{A}}_2$. Then

$$\underline{\underline{V}} \wedge \underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}} \not\geq \underline{\underline{A}}_2\underline{\underline{A}}_2,$$

and so $\underline{\underline{V}} \wedge \underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}}$ is nilpotent by 5.3.12. Thus we can write

5.3.15 $\quad V.\underline{\underline{A}}_2\underline{\underline{A}}(H(2)) \geq \gamma_r(H(2))$

for some $r \in I$, where $V = \underline{\underline{V}}\mu_{H(2)}$, and therefore

$$V \geq [V.\underline{\underline{A}}_2\underline{\underline{A}}(H(2)),\underline{\underline{A}}(H(2))] = [V,\underline{\underline{A}}(H(2))]$$

$$\geq [\gamma_r(H(2)),\underline{\underline{A}}(H(2))] \geq [\underline{\underline{A}}\underline{\underline{A}}(H(2)),(r-2)H(2)]$$

using several applications of 1.2.2(iii).

Thus by 5.2.10 $V \geq [\underline{\underline{A}}_2\underline{\underline{A}}(H(2)),(r-1)H(2)]$, and using 5.3.15 again we have

$$V \geq [V.\underline{\underline{A}}_2\underline{\underline{A}}(H(2)),(r-1)H(2)] = [V,(r-1)H(2)][\underline{\underline{A}}_2\underline{\underline{A}}(H(2)),(r-1)H(2)]$$

$$\geq [\gamma_r(H(2)),(r-1)H(2)] = \gamma_{2r-1}(H(2)),$$

and $\underline{\underline{V}}$ is nilpotent as claimed.

We now consider the case $\underline{\underline{V}} \not\leq \underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}}$ and $\underline{\underline{V}} \geq \underline{\underline{A}}_2\underline{\underline{A}}_2$. By 5.3.14 $V \cap \underline{\underline{A}}_2\underline{\underline{A}}(H(2)) \neq \{1\}$, and also $V \cap \underline{\underline{A}}_2\underline{\underline{A}}(H(2)) \in Id^{\#}(\underline{\underline{A}}_2\underline{\underline{A}}(H(2))$. Therefore, by 5.3.11 $V \cap \underline{\underline{A}}_2\underline{\underline{A}}(H(2)) \geq [\underline{\underline{A}}_2\underline{\underline{A}}(H(2)),sH(2)]$ for some $s \in I$. On the other hand, as we saw in the proof of 5.3.14,

$$V.\underline{\underline{A}}_2\underline{\underline{A}}(H(2)) \geq [\underline{\underline{A}}_2\underline{\underline{A}}(H(2)),k\,H(2)]$$

for some $k \in I$. Thus

$$\begin{aligned} V &\geq [V.\underline{\underline{A}}_2\underline{\underline{A}}(H(2)),s\,H(2)] = [V,s\,H(2)][\underline{\underline{A}}_2\underline{\underline{A}}(H(2)),s\,H(2)] \\ &\geq [\underline{\underline{A}}_2\underline{\underline{A}}_2(H(2)),(k+s)H(2)] \end{aligned}$$

and so $\underline{\underline{V}} \leq [\underline{\underline{A}}_2\underline{\underline{A}}_2,r\,\underline{\underline{E}}]$ where $r = k + s$.

We showed in 5.3.12 that $\underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \wedge \underline{\underline{A}}_2\underline{\underline{A}} \not\leq [\underline{\underline{A}}_2\underline{\underline{A}}_2,s\,\underline{\underline{E}}]$ for any $s \in I$, so it remains to show that $\underline{\underline{A}}_2\underline{\underline{T}}_2 \wedge \underline{\underline{T}}_2\underline{\underline{A}}_2 \not\leq [\underline{\underline{A}}_2\underline{\underline{A}},r\,\underline{\underline{E}}]$ for any $r \in I$. But if it were this would imply that $[[x,y,z],[u,v],x_1,\ldots,x_r]$ is a law in $H(2)$ for some $r \in I$. But this would contradict 5.2.3.

## 5.4 The Proof of 5.3.4

The culmination of the work in this section will be a proof of 5.3.4. This proof breaks up into two parts, namely, a proof that $w^{(i)}$ is special for the three cases treated in 5.3.4, and then a proof that $w^{(i)}$ is non-trivial for at least one value of $i \in I^+$. The methods used rely heavily on those of Brooks (see section 2.4 of [6]), but none of his proofs are used as they are not applicable here.

When no confusion will arise we will let $H$ represent $H(p)$ and $\underline{\underline{h}} = \{h_i : i \in I\}$ be a free generating set for $H$. We will still use the convention of letting $G$ represent $H(p)$ for any primes $p$, or $K$, when we want to make general statements about all these groups.

From the definition 5.3.3 it is clear that for each (fixed) $i \in I^+$ the mapping of $\underline{\underline{T}}_p(G)$ into itself given by $w \to w^{(i)}$ for all $w \in \underline{\underline{T}}_p(G)$ is an endomorphism of $\underline{\underline{T}}_p(G)$, The final objective, then, will be to describe the effect of these endomorphisms of $\underline{\underline{T}}_p(G)$ on the members of the basis of $\underline{\underline{T}}_p(G)$. This description is quite involved, and the necessary information will be given in the following lemmas.

The first two results give some identities that will often be used without explicit reference.

5.4.1 <u>Lemma</u>: (a) Let $u,v,w,x,y \in H$. Then

(i) $[[u,v,pw],[x,y]] = 1$

(ii) $[[u,pw,v],[x,y]] = [[v,pw,u],[x,y]$.

(b) Let $u,v,w,x \in H(2)$. Then

(i) $[[u,v,2w],x^2] = 1$

(ii) $[[u,2w,v],x^2] = [[v,2w,u],x^2]$.

<u>Proof</u>: (a)(i) This result follows since $[u,v,pw] \in \underline{\underline{A}}_p\underline{\underline{A}}_p(H)$.

(ii) $[[u,pw,v],[x,y]] = [[v,pw,u]t,[x,y]]$ where $t \in \underline{\underline{A}}_p\underline{\underline{A}}_p(H)$

$= [[v,pw,u],[x,y]]$.

The results for (b) are proved in the same way.

5.4.2 <u>Lemma</u>: Let $a \in G$, $b \in \gamma_3(G)$. Then

$$(ab)^p = a^p[b,(p-1)a].$$

<u>Proof</u>: It is easy to prove by induction on $r$ for $r \geq 1$, that

$$(ab)^r = a^r \prod_{i=1}^{r} [b,(i-1)a]^{\binom{r}{i}}.$$

The result follows by putting $r = p$ since $\binom{p}{i} \equiv 0 \bmod p$ for all $i \in \{1,\ldots,p-1\}$.

5.4.3 <u>Notation</u>: For each integer $i > 0$ and each degree function $\delta$ on $C(G)$ with $g_i \in \operatorname{supp}\delta$ let $\delta^{(i)}$ be the degree function on $C(G)$ defined by

$$\delta^{(i)}(g_1) = \delta^{(i)}(g_2) = \delta^{(i)}(g_3) = 1$$

$$\delta^{(i)}(g_{i+3}) = \delta(g_i) - 1$$

$$\delta^{(i)}(g_j) = \delta(g_{j-3}) \quad \text{for all } j \geq 4, \ j \neq i + 3,$$

and

$$\delta^{(i)}(a) = 0 \quad \text{for all } a \in C(G)\backslash \underline{g}.$$

A straightforward commutator calculation verifies the following results.

5.4.4 <u>Lemma</u>: Let $u \in \underline{\underline{T}}_p(H)$ such that $u = [[a,b,\delta],[h_j,h_m]]$ where $(a,b,\delta) \in B(\underline{\underline{h}};p)\backslash \underline{\underline{a}}(\underline{\underline{h}})$. Then

Then

$$u^{(i)} = \begin{cases} [[h_2,h_1,h_3,\delta^{(i)}],[h_{j+3},h_{m+3}]] & \text{if } a = h_i \\ [[h_2,h_1,h_3,\delta^{(i)}],[h_{j+3},h_{m+3}]]^{-1} & \text{if } b = h_i \\ 1 \ \text{otherwise} & \end{cases}$$

We remark here that if $u$ is as above, then $u^{(i)}$ is a special element of $\underline{\underline{T}}_p(H)$. The following lemma shows that if $u$ is also a basis element then $u^{(i)}$ is either trivial or is again a basis element.

5.4.5 <u>Lemma</u>: Let $u = [[h_{i_1},h_{i_2},\delta],[h_j,h_m]]$ be a basis element in $\underline{\underline{T}}_p(H)$ and let $\{k,\ell\} = \{1,2\}$. Then for both $k = 1$ and $k = 2$

(i) $\delta(h_{i_\ell}) < p \Rightarrow u^{(i_k)}$ is also a basis element of $\underline{\underline{T}}_p(H)$,

(ii) $\delta(h_{i_\ell}) = p \Rightarrow u^{(i_k)} = 1$ .

Proof: The proof follows immediately from 5.4.4 and 5.4.1(i).

5.4.6 Lemma: Let $u \in \underline{\underline{T}}_p(H)$ such that $u = [[h_i,h_j],[h_m,h_n]]$ where $i,j,m$ and $n$ are distinct integers. Then

(i) $u^{(i)} = [[h_2,h_1,h_3,h_{j+3}],[h_{m+3},h_{n+3}]$

(ii) $u^{(j)} = [[h_2,h_1,h_3,h_{i+3}],[h_{m+3},h_{n+3}]]^{-1}$

(iii) $u^{(m)} = [[h_2,h_1,h_3,h_{n+3}],[h_{i+3},h_{j+3}]]^{-1}$

(iv) $u^{(n)} = [[h_2,h_1,h_3,h_{m+3}],[h_{i+3},h_{j+3}]]$

(v) $u^{(s)} = 1$ for $s \in I^+ \setminus \{i,j,m,n\}$.

Also if $u$ is a basis element of $\underline{\underline{T}}_p(H)$ then for all $i \in I^+$, $u^{(i)}$ is trivial or $u^{(i)}$ or $(u^{(i)})^{-1}$ is a basis element and is pspecial.

5.4.7 Lemma: Let $u \in \underline{\underline{T}}_p(H)$ such that $u = [[h_i,h_j],[h_i,h_m]]$.

Then

(i) $u^{(i)} = [[h_2,h_1,h_3,h_{j+3}],[h_{i+3},h_{m+3}]][[h_2,h_1,h_3,h_{m+3}],[h_{i+3},h_{j+3}]]^{-1}$

(ii) $u^{(j)} = [[h_2,h_1,h_3,h_{i+3}],[h_{i+3},h_{m+3}]]^{-1}$

(iii) $u^{(m)} = [[h_2,h_1,h_3,h_{i+3}],[h_{i+3},h_{j+3}]]$

(iv) $u^{(s)} = 1$ for $s \in I^+ \setminus \{i,j,m\}$.

Also, if $u$ is a basis element of $\underline{\underline{T}}_p(H)$ and $u^{(n)}$ is non-trivial then $u^{(n)}$ is a basis element or is a product of basis elements, and $u^{(n)}$ is special for all $n \in I^+$.

5.4.3 Lemma: Let $u \in \underline{\underline{A}}_p\underline{\underline{A}}_p(H(p))$ for $p \neq 2$, such that $u = [h_i^p,[h_j,h_k]]$. Then

$$u^{(i)} = [[h_2,h_1,h_3,(p-1)h_{i+3}],[h_{j+3},h_{k+3}]],$$

$$u^{(s)} = 1 \quad \text{for all } s \in I^+ \setminus \{i\}.$$

5.4.9 Lemma: Let $u \in \underline{\underline{A}}_2\underline{\underline{A}}(H(2))$ such that $u = [h_i,h_j]^2$. Then

(i) $u^{(i)} = [[h_2,h_1,h_3,h_{j+3}],[h_{i+3},h_{j+3}]]$

(ii) $u^{(j)} = [[h_2,h_1,h_3,h_{i+3}],[h_{i+3},h_{j+3}]]$

(iii) $u^{(s)} = 1$ for all $s \in I^+ \setminus \{i,j\}$.

We now look at the effect of these endomorphisms on basis elements in $\underline{\underline{A}}_2\underline{\underline{A}}_2(K)$. The proofs are straightforward commutator calculations and are not given here.

5.4.10 Lemma: Let $(a,b,\delta) \in B(\underline{\underline{k}};2)$ and let $u = [[a,b,\delta],k_j^2]$ be a basis element in $\underline{\underline{A}}_2\underline{\underline{A}}_2(K)$. Then

$$u^{(i)} = \begin{cases} [[k_2,k_1,k_3,\delta^{(i)}],k_j^2] & \text{if } a = k_i \\ [[k_2,k_1,k_3,\delta^{(i)}],k_j^2]^{-1} & \text{if } b = k_i \\ 1 \text{ otherwise} \end{cases}$$

Further, $u^{(i)}$ is special for all $i \in I^+$, and if $u^{(i)}$ is non-trivial then $u^{(i)}$ is a basis element of $\underline{\underline{A}}_2\underline{\underline{A}}_2(K)$.

5.4.11 Lemma: Let $u$ be a basis element in $\underline{\underline{A}}_2\underline{\underline{A}}_2(K)$ such that $u = [k_i^2,k_j^2]$. Then

(i) $u^{(i)} = [[k_2,k_1,k_3,k_{i+3}],k_j^2]$

(ii) $u^{(j)} = [[k_2,k_1,k_3,k_{j+3}],k_i^2]$

(iii) $u^{(s)} = 1$ for all $s \in I^+ \setminus \{i,j\}$.

5.4.12 Lemma: Let $u = k_i^4$ for some $i \in I^+$. Then $u^{(i)} = [[k_2,k_1,k_3,k_{i+3}],k_{i+3}^2]$, and $u^{(j)} = 1$ for all $j \in I^+ \setminus \{i\}$.

With the next lemma we will be a good way towards the proof of 5.3.4.

5.4.13 Lemma: (i) For all $w \in \underset{=}{A}_p\underset{=}{A}_p(H(p))$ for $p \neq 2$, and all $i \in I^+$, $w^{(i)}$ is special.

(ii) For all $w \in \underset{=}{A}_2\underset{=}{A}(H(2))$, and all $i \in I^+$, $w^{(i)}$ is special.

(iii) For all $w \in \underset{=}{A}_2\underset{=}{A}_2(K)$, and all $i \in I^+$, $w^{(i)}$ is special.

Proof: (i) For $w = 1$ there is nothing to prove, so assume $w \neq 1$ and let $w$ be expressed in normal form by $w = b_1^{e_1} \ldots b_t^{e_t}$ where $b_j \in \underset{=}{A}_p\underset{=}{A}_p(H(p))$ for $j \in \{1,\ldots,t\}$ and $t \geq 1$. Then for any $i \in I^+$,

$$w^{(i)} = (b_1^{(i)})^{e_1} \ldots (b_t^{(i)})^{e_t}.$$

But lemmas 5.4.4 - 5.4.8 together with 5.2.7 show that if $b$ is a basis element in $\underset{=}{A}_p\underset{=}{A}_p(H(p))$, then $b^{(i)}$ is special for all $i \in I^+$. Thus $w^{(i)}$ is a product of special elements and so is special.

(ii) The proof is similar to part (i). The relevant lemmas here are 5.4.4 - 5.4.7, 5.4.9 and 5.2.4.

(iii) The proof is again similar to part (i). The relevant lemmas are 5.4.10 - 5.4.12 and 5.2.8.

It remains to show in each of these cases that if $w$ is non-trivial then so is $w^{(i)}$ for at least one value of $i$. The proof is in several steps and we use the endomorphisms described in the following definition. Here again $G$ represents $H(p)$ or $K$.

5.4.14 Definition: For each $v,w \in G$ and $i \in I^+$ let $\sigma(v,w,i)$ be the endomorphism of $G$ induced by the map $\bar{\sigma}(v,w,i) : \underset{=}{g} \to G$ defined by

$$g_1\, \bar{\sigma}(v,w,i) = v$$
$$g_2\, \bar{\sigma}(v,w,i) = g_i$$
$$g_3\, \bar{\sigma}(v,w,i) = w$$
$$g_j\, \bar{\sigma}(v,w,i) = g_{j-3} \quad \text{for all } j \in I^+ \setminus \{1,2,3\}.$$

We again check the action of these endomorphisms on the basis elements of the relevant subgroups of $H(p)$ and $K$. The result is the following series of lemmas. These lemmas are proved by straightforward commutator calculations using the previous lemmas and the identities in Chapter 1. The proofs are therefore omitted. Here again $H$ represents $H(p)$ for some prime $p$.

5.4.15 Lemma: Let $u$ be a basis element in $\underset{=}{A}_p\underset{=}{A}_p(H)$ such that $u = [[h_i,h_j,\delta],[h_m,h_n]]$ where $[h_i,h_j,\delta] \in \delta_3(H)$. Then for all $v,w \in H$

$$u^{(i)}\sigma(v,w,i)u^{(j)}\sigma(v,w,j) = [u,v,w].$$

5.4.16 Lemma: Let $u$ be a basis element in $\underset{=}{A}_p\underset{=}{A}_p(H)$ such that $u = [[h_i,h_j],[h_m,h_n]]$ where $i,j,m$ and $n$ are distinct integers. Then for all $v,w \in H$

$$u^{(i)}\sigma(v,w,i)u^{(j)}\sigma(v,w,j)u^{(m)}\sigma(v,w,m)u^{(n)}\sigma(v,w,n) = [u,v,w]$$

5.4.17 Lemma: Let $u$ be a basis element in $\underset{=}{A}_p\underset{=}{A}_p(H)$ such that

(i) $u = [[h_i,h_j],[h_i,h_m]]$

(ii) $u = [[h_i,h_j],[h_j,h_m]]$

(iii) $u = [[h_i,h_j],[h_m,h_j]]$.

Then for all $v,w \in H$

$$u^{(i)}\sigma(v,w,i)u^{(j)}\sigma(v,w,j)u^{(m)}\sigma(v,w,m) = [u,v,w].$$

5.4.18 Lemma: Let $u = [h_i^p,[h_j,h_m]]$ be a basis element in $\underset{=}{A}_p\underset{=}{A}_p(H(p))$ for $p \neq 2$. Then for all $v,w \in H(p)$,

$$u^{(i)}\sigma(v,w,i) = [u,v,w].$$

5.4.19 Lemma: Let $u$ be a basis element in $\underset{=}{A}_2\underset{=}{A}(H(2))$ such that $u = [h_i,h_j]^2$. Then for all $v,w \in H(2)$,

$$u^{(i)}\sigma(v,w,i)u^{(j)}\sigma(v,w,j) = [u,v,w].$$

5.4.20 Lemma: Let $u$ be a basis element in $\underset{=}{A}_2\underset{=}{A}_2(K)$ such that $u = [[k_i,k_j,\delta],k_m^2]$. Then for all $v,w \in K$,

$$u^{(i)}\sigma(v,w,i)u^{(j)}\sigma(v,w,j) = [u,v,w].$$

5.4.21 Lemma: Let $u = [k_i^2,k_j^2]$ be a basis element in $\underset{=}{A}_2\underset{=}{A}_2(K)$. Then for all $v,w \in K$

$$u^{(i)}\sigma(v,w,i)u^{(j)}\sigma(v,w,j) = [u,v,w].$$

5.4.22 Lemma: Let $u = k_i^4$. Then for all $v,w \in K$, $u^{(i)}\sigma(v,w,i) = [u,v,w]$.

Before summarizing these lemmas we make the following definition.

5.4.23 Definition: Let $b$ be a basis element of $\underset{=}{T}_p(G)$. Then the set of entries of $b$, denoted by $E(b)$ is defined by

$$E(b) = \min\{\underset{=}{g}' \subseteq \underset{=}{g} : b \in gp\ \underset{=}{g}'\}.$$

If $w$ is a non-trivial element of $\underset{=}{T}_p(G)$ expressed in normal form by $w = b_1^{e_1}\ldots b_t^{e_t}$ then the set of entries of $w$, denoted by $E(w)$ is defined by $E(w) = \bigcup_{j=1}^{t} E(b_j)$. In addition let $E(1)$ be $\emptyset$, and for any $w_1,\ldots,w_m \in \underset{=}{T}_p(G)$ denote $\bigcup_{i=1}^{m} E(w_i)$ by $E(w_1,\ldots,w_m)$.

5.4.24 Lemma: (i) Let $u$ be a basis element in $\underset{=}{A}_p\underset{=}{A}_p(H(p))$, where $p \neq 2$, and let $J$ be a finite subset of $I^+$ such that if $h_i \in E(u)$, then $i \in J$. Then

$$\prod_{j \in J} (u^{(j)}\sigma(v,w,j)) = [u,v,w].$$

(ii) Let $u$ be a basis element in $\underset{=}{A}_2\underset{=}{A}(H(2))$, and let $J$ be a finite subset of $I^+$ such that if $h_i \in E(u)$ then $i \in J$. Then

$$\prod_{j \in J} (u^{(j)}\sigma(v,w,j)) = [u,v,w].$$

(iii) Let $u$ be a basis element in $\underset{=}{A}_2\underset{=}{A}_2(K)$, and let $J$ be a finite subset of $I$ such that $g_i \in E(u) \Rightarrow j \in J$. Then

$$\prod_{j \in J} (u^{(j)}\sigma(v,w,j)) = [u,v,w].$$

<u>Proof</u>: The proof is immediate from the previous lemmas.

The proof of 5.3.4 will follow from the next lemma.

5.4.25 <u>Lemma</u>: (i) For all $w \in \underset{=}{A}_p\underset{=}{A}_p(H(p))$, for $p \neq 2$, and all $v,s \in H(p)$, $[w,v,s] \in \langle w^{(i)} | i \in I^+\rangle$.

(ii) For all $w \in \underset{=}{A}_2\underset{=}{A}(H(2))$ and all $v,s \in H(2)$,

$$[w,v,s] \in \langle w^{(i)} | i \in I^+\rangle.$$

(iii) For all $w \in \underset{=}{A}_2\underset{=}{A}_2(K)$ and all $v,s \in K$,

$$[w,v,s] \in \langle w^{(i)} | i \in I^+\rangle.$$

<u>Proof</u>: We give the proof of (i). The proofs of (ii) and (iii) are exactly the same except that we are working in different groups.

For $w = 1$ there is nothing to prove. So assume $w \neq 1$ and let $w$ be expressed as a product of basis elements in $\underset{=}{A}_p\underset{=}{A}_p(H(p))$ by $w = b_1^{e_1} \ldots b_t^{e_t}$ where $t \geq 1$. Let $J = \{i \in I : h_i \in E(w)\}$. Then for any $v,s \in H(p)$ we have

$$\prod_{j\in J}(w^{(j)}\sigma(v,s,j)) = \prod_{j\in J}((\prod_{k=1}^{t} b_k^{e_k})^{(j)}\sigma(v,s,j))$$

$$= \prod_{j\in J}((\prod_{k=1}^{t}(b_k^{(j)})^{e_k})\sigma(v,s,j))$$

$$= \prod_{j\in J}(\prod_{k=1}^{t}(b_k^{(j)}\sigma(v,s,j))^{e_k})$$

$$= \prod_{k=1}^{t}(\prod_{j\in J}(b_k^{(j)}\sigma(v,s,j)))^{e_k}$$

$$= \prod_{k=1}^{t}[b_k,v,s]^{e_k} \quad \text{by } 5.4.24$$

$$= [\prod_{k=1}^{t} b_k^{e_k},v,s] \quad \text{by } 1.2.3(\text{i})$$

$$= [w,v,s].$$

Hence $[w,v,s] \in \langle w^{(j)} | j \in J\rangle$, and 5.4.25 follows.

<u>Proof of 5.3.4</u>: We have shown in each case that $w^{(i)}$ is special for all $i \in I^+$, and in view of 5.4.25 it is now sufficient to show that if $w \neq 1$ there exist $v$ and $s$ such that $[w,v,s] \neq 1$. We shall give an explicit proof for part (i) and just note any differences for parts (ii) and (iii).

(i) Let $w$ be expressed in normal form by $w = b_1^{e_1}\dots b_t^{e_t}$. Choose $u,v \in \underline{\underline{h}}(p)\setminus E(w)$ such that $u > v > h_i$ for all $h_i \in E(w)$.

Then $$[w,u,v] = [\prod_{j=1}^{t} b_j^{e_j},u,v]$$
$$= \prod_{j=1}^{t}[b_j,u,v]^{e_j}.$$

We now look at $[b_j,u,v]$ for the different types of basis elements. If $b_j = [[h_{i_1},h_{i_2},\delta],[h_j,h_k]]$, where $[h_{i_1},h_{i_2},\delta] \in \gamma_3(H(p))$, then

$[b_j,u,v] = [[h_{i_1},h_{i_2},\delta'],[h_j,h_k]]$ where $\delta'(h_i) = \delta(h_i)$ for $h_i \in E(b_j)$

and $\delta'(u) = \delta'(v) = 1$, and because of the choice of $u$ and $v$, $[b_j,u,v]$ is also a basic commutator element in $\underline{\underline{T}}_p(H(p))$.

If

$$b_j = [[h_i,h_k],[h_\ell,h_m]] \text{ then}$$
$$[b_j,u,v] = [[h_i,h_k,u,v],[h_\ell,h_m]][[h_\ell,h_m,u,v],[h_i,h_k]]^{-1}$$

and thus $[b_j,u,v]$ is written as a product of basis elements.

If $b_j = [h_i^p,[h_k,h_\ell]]$, then

$$[b_j,u,v] = [[h_i,\ u,(p-1)h_i,v],[h_k,h_\ell]]$$
$$= [[v,ph_i,u],[h_k,h_\ell]]$$

which is a basis element in $\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p))$.

Thus, $[w,u,v]$ can be written as a product of non-trivial basis elements in $\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p))$ which are distinct as long as $b_1,\ldots,b_t$ are distinct. It follows that $[w,u,v] \neq 1$.

(ii) The proof for part (ii) obviously follows the same pattern as that for part (i). In this case the basis elements can take the first two forms considered in part (i), but not the last form. In this case we have also to consider $b_j = [h_i,h_j]^2$. Then $[b_j,u,v] = [[h_i,h_j,u,v],[h_i,h_j]]$ which is a basis element of $\underline{\underline{A}}_2\underline{\underline{A}}(H(2))$ and the conclusion follows as before.

(iii) In this case there are several different forms of basis element to consider. If $b_j = [[k_{i_1},k_{i_2},\delta],k_\ell^2]$ then $[b_j,u,v] = [[k_{i_1},k_{i_2},\delta'],k_\ell^2]$ where $\delta'(k_i) = \delta(k_i)$ for $k_i \in E(b_j)$ and $\delta'(u) = \delta'(v) = 1$. But then $[b_j,u,v]$ is again a basis element.

If $b_j = [k_i^2,k_\ell^2]$, then

$$[b_j,u,v] = [[k_i,u,v,k_i],k_\ell^2][[k_\ell,u,v,k_\ell],k_i^2]$$
$$= [[v,2k_i,u],k_\ell^2][[v,2k_\ell,u],k_i^2],$$

which is a product of basis elements in $\underline{\underline{A}}_2\underline{\underline{A}}_2(K)$.

If

$$b_j = k_i^4, \text{ then}$$

$$[b_j,u,v] = [[k_i,u,v,k_i],k_i^2]$$

$$= [[v,2k_i,u],k_i^2]$$

which is again a basis element in $\underline{\underline{A}}_2\underline{\underline{A}}_2(K)$ and the conclusion follows as in part (i).

## 5.5 The Proofs of 5.3.5, 5.3.8 and 5.3.9

Many of the methods used in this section originate in the thesis of R.A. Bryce (see [10]) and they have also been employed by Brooks [6] and [8]). However, while there is a heavy "borrowing" of methods and extending of these methods, there is no "borrowing" of results as both Bryce and Brooks worked in metabelian groups.

We begin by stating a simple result that will make the proofs of 5.3.5, 5.3.8 and 5.3.9 a little easier. First we require the following definition.

5.5.1 Definition: Let $w$ be a non-trivial element of $\underline{\underline{T}}_p(G)$ expressed in normal form by $w = b_1^{e_1} \ldots b_t^{e_t}$. Then $w$ is called homogeneous if, and only if,

$$E(b_1) = E(b_2) = \ldots = E(b_t).$$

Any non-trivial element $w \in \underline{\underline{T}}_p(G)$ is the product of its homogeneous parts; that is $w = w_1 \ldots w_s$ where $w_1, \ldots, w_s$ are non-trivial homogeneous elements of $\underline{\underline{T}}_p(G)$ with $E(w_i) \neq E(w_j)$ if $i \neq j$.

5.5.2 Lemma: If $w$ is a non-trivial element of $\underline{\underline{T}}_p(G)$ then $\langle w \rangle \geq \langle w' \rangle$ for every homogeneous part $w'$ of $w$.

Proof: The lemma is a special case of 33.45 of [17].

If $w$ is a non-trivial special element of $\underline{\underline{T}}_p(G)$ it is clear that the homogeneous parts of $w$ are themselves special. From 5.5.2 it will be sufficient to prove 5.3.5, 5.3.8 and 5.3.9 for non-trivial homogeneous special elements.

The proofs of 5.3.5 and 5.3.9 will be done together as special elements in $\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p))$, for $p \neq 2$, and in $\underline{\underline{A}}_2\underline{\underline{A}}(H(2))$ have the same form. The proofs will be preceded by a number of preliminary lemmas, and it is in these that the methods of Bryce and Brooks are used. In these lemmas we again write $H$ for $H(p)$.

5.5.3 Lemma: For all $u,v \in H$, $w \in \underline{\underline{A}}\underline{\underline{A}}(H)$, $i \in I^+$, $[w,(uv)^i] = [w,u^i v^i]$.

Proof: For some $c \in \underline{\underline{A}}(H)$, $(uv)^i = u^i v^i c$, so $[w,(uv)^i] = [w,u^i v^i c] = [w,u^i v^i]^c [w,c] = [w,u^i v^i]$.

5.5.4 Lemma: Let $w \in \underline{\underline{T}}_p(H)$, and let $V$ be a fully invariant subgroup of $H$. If for all $v \in H$, $[w,v^i] \in V$ for some $i \in I^+$, then

$$\text{g.c.d}(i,p) = 1 \Rightarrow [w,v] \in V \text{ for all } v \in V.$$

Proof: There exist integers $a$ and $b$ such that $ai + bp = 1$, and since $\underline{\underline{T}}_p(H)$ is abelian of exponent $p$ it follows that

$$\begin{aligned}[w,v] &= [w,v^{ai+bp}] \\ &= [w,(v^a)^i (v^b)^p] \\ &= [w,(v^a)^i] \in V.\end{aligned}$$

5.5.5 Lemma: Let $V$ be a fully invariant subgroup of $H$ and let $J$ be a finite subset of $I$. If for fixed elements $a_1,\ldots,a_m \in [\gamma_3(H),\gamma_2(H)]$, where $m < p$, $b_{ij} \in \gamma_3(H)(0 \leq i \leq m, j \in J)$, $u_j \in H(j \in J)$, and for all $v \in H$,

$$\prod_{i=1}^{m} [a_i,v^i] \prod_{j \in J} \{[b_{oj},[v,u_j]][b_{ij},[v,u_j],v^i]\} \in V,$$

then for all $x_1,y_1,\ldots,x_m,y_m,z \in V$,

$$\prod_{j \in J} [b_{mj},[v,u_j],x_m,y_m,\ldots,x_1,y_1] \in V \text{ and}$$

$$[a_m,x_m,y_m,\ldots,x_1,y_1,z] \in V.$$

Proof: The proof is by induction on $m$. For $m = 0$ there is nothing to prove. Assume the assertion true for $m' < m$. Then for any $x_m \in H$ we have

$$\prod_{i=1}^{m} [a_i,(vx_m)^i] \prod_{j \in J}\{[b_{oj},[vx_m,u_j]][b_{ij},[vx_m,u_j],(vx_m)^i]\} \in V$$

for all $v \in H$, and hence by expanding this, using 5.5.3, and applying the original hypothesis, we have

$$5.5.6 \qquad \prod_{i=1}^{m} \{[a_i,x_m^i,v^i] \prod_{j \in J}\{[b_{ij},[v,u_j],x_m^i][b_{ij},[v,u_j],x_m^i,v^i]$$
$$\times [b_{ij},[x_m,u_j],v^i][b_{ij},[x_m,u_j],x_m^i,v^i]\}\} \in V$$

for all $v \in V$. If we replace $v$ by $vy_m$, for any $y_m \in H$, the product is still in $V$, and if we expand and apply 5.5.6 we get

$$5.5.7 \qquad \prod_{i=1}^{m} [a_i,x_m^i,y_m^i,v^i] \prod_{j \in J} [b_{ij},[v,u_j],x_m^i,y_m^i][b_{ij},[v,u_j],x_m^i,y_m^i,v^i]$$
$$\times [b_{ij},[x_m,u_j],v^i,y_m^i][b_{ij},[x_m,u_j],x_m^i,y_m^i,v^i]$$
$$\times [b_{ij},[y_m,u_j],x_m^i,v^i][b_{ij},[y_m,u_j],x_m^i,y_m^i,v^i]\} \in V$$

for all $v \in H$. From 5.5.6 we also have

$$5.5.8 \quad \prod_{i=1}^{m} [a_i,x_m^i,y_m^i,v]^{-1} \prod_{j\in J} \{[b_{ij},[x_m,u_j],y_m^i,v]^{-1}$$

$$[b_{ij},[x_m,u_j],x_m^i,y_m^i,v]^{-1}[b_{ij},[y_m,u_j], x_m^i,v]^{-1}$$

$$[b_{ij},[y_m,u_j],x_m^i,y_m^i,v]^{-1}\} \in V$$

for all $v \in H$.

If we multiply the expressions in 5.5.7 and 5.5.8 and put

$$A_i = [a_{i+1},x_m^{i+1},y_m^{i+1}] \prod_{j\in J} \{[b_{i+1,j},[x_m,u_j],y_m^{i+1}][b_{i+1,j},[x_m,u_j],x_m^{i+1},y_m^{i+1}]$$

$$\times [b_{i+1,j},[y_m,u_j],x_m^{i+1}][b_{i+1,j},[y_m,u_j],x_m^{i+1},y_m^{i+1}]\}$$

for $i \in \{0,1,\dots,m+1\}$,

$$B_{oj} = \prod_{i=1}^{m} [b_{ij},x_m^i,y_m^i] \quad \text{for } j \in J$$

$$B_{ij} = [b_{i+1,j},x_m^{i+1},y_m^{i+1}] \quad \text{for } i \in \{1,\dots,m-1\},\ j \in J,$$

and use the identity $[w,v]^{-1}[w,v^{i+1}] = [w,v^i]^v$, we get

$$\prod_{i=1}^{m-1} [A_i,v^i]^v \prod_{j\in J} \{[B_{oj},[v,u_j]]^v[B_{ij},[v,u_j],v^i]^v\} \in V$$

for all $v \in H$. Since $V$ is normal in $H$ we have

$$\prod_{i=1}^{m-1} [A_i,v^i] \prod_{j\in J} \{[B_{oj},[v,u_j]][B_{ij},[v,u_j],v^i]\} \in V$$

for all $v \in H$, and therefore by the inductive hypothesis

$$\prod_{j\in J} [B_{m-1,j},[v,u_j],x_{m-1},y_{m-1},\dots,x_1,y_1] \in V \quad \text{for all } x_{m-1},y_{m-1},\dots,$$

$x_1,y_1 \in H$ and $[A_{m-1},x_{m-1},y_{m-1},\dots,x_1,y_1,z,] \in V$ for all $x_{m-1},y_{m-1},\dots,x_1,y_1,z \in H$. But $B_{m-1,j} = [b_{mj},x_m^m,y_m^m]$, so that we have

$$\prod_{j\in J} [b_{mj},[v,u_j],x_m^m,y_m^m,x_{m-1},y_{m-1},\dots,x_1,y_1] \in V, \quad \text{and therefore}$$

by 5.5.4,

5.5.9 $\prod_{j\in J} [b_{mj},[v,u_j],x_m,y_m,\ldots,x_1,y_1] \in V$ for all $x_1,y_1,\ldots,x_m,y_m \in H$. Also,

$$A_{m-1} = [a_m,x_m^m,y_m^m] \prod_{j\in J} \{[b_{mj},[x_m,u_j],y_m^m][b_{mj},[x_m,u_j],x_m^m,y_m^m]$$

$$\times [b_{mj},[y_m,u_j],x_m^m][b_{mj},[y_m,u_j],x_m^m,y_m^m]\},$$

so that $[A_{m-1},x_{m-1},y_{m-1},\ldots,x_1,y_1,z] \in V$ implies, together with 5.5.9, that

$$[a_m,x_m^m,y_m^m,x_{m-1},y_{m-1},\ldots,x_1,y_1,z] \in V, \text{ and therefore by 5.5.4,}$$

$$[a_m,x_m,y_m,\ldots,x_1,y_1,z] \in V \text{ for all } x_m,y_m,\ldots,x_1,y_1,z \in H.$$

In the subsequent lemmas we will use the following notation. Let $G$ be any group and let $U$ be normal in $G$. The subgroups $U_i$ of $G$ for $i \in I^+$ are defined by

$$U_i/U = Z_i(G/U)$$

where $Z_i(G/U)$ is the i-th term of the upper central series of $G/U$. Note that if $g \in G$, $[g,b_1,\ldots,b_r] \in U$ for all $b_1,\ldots,b_r$ in $G$ if, and only if, $g \in U_r$.

5.5.10 <u>Lemma</u>: Let $V$ be a fully invariant subgroup of $H$, $J$ a finite subset of $I$, and let

$$\prod_{i=1}^{m} [a_i,iv] \prod_{j\in J} \{[b_{oj},[v,u_j]][b_{ij},[v,u_j],iv]\} \in V \quad \text{where}$$

$m \in \{1,\ldots,p-1\}$, $a_1,\ldots,a_m \in [\gamma_3(H),\gamma_2(H)]$, $b_{ij} \in \gamma_3(H)$

$(0 \leq i \leq m,\ j \in J)$

$u_j \in \underline{\underline{h}}(j \in J)$, and $v \in \underline{\underline{h}}\setminus E(a_i,b_{oj},b_{ij} : 1 \leq i \leq m, j \in J) \cup \{u_j : j \in J\}$.

Then

$$\prod_{j\in J} [b_{mj},[v,u_j]] \in V_{2m} \text{ and } a_m \in V_{2m+1}.$$

<u>Proof</u>: Using the identity $[t,ku] = \prod_{i=1}^{k} [t,u^i]^{(-1)^{k-i}\binom{k}{i}}$ for $t \in \gamma_3(H)$, $u \in H$, we have

$$\prod_{i=1}^{m} [A_i,v^i] \prod_{j \in J} \{[B_{oj},[v,u_j]][B_{ij},[v,u_j],v^i]\} \in V$$

where each $A_i$ is a linear combination of $a_1,\ldots,a_m$, and $A_m = a_m$, and each $B_{ij}$ is a linear combination of $b_{1j},\ldots,b_{mj}$, and $B_{mj} = b_{mj}$ for all $j \in J$. Also $B_{oj} = b_{oj}$ for all $j \in J$. Now, for any $h \in H$ there is an endomorphism $\theta$ of $H$ such that $A_i\theta = A_i$, $B_{ij}\theta = B_{ij}$, $u_j\theta = u_j$ and $v\theta = h$. So it follows that

$$\prod_{i=1}^{m} [A_i,h^i] \prod_{j \in J} \{[B_{oj},[h,u_j]][B_{ij},[h,u_j],h^i]\} \in V$$

for all $h \in H$. Thus by 5.5.5 $\prod_{j \in J} [B_{mj},[v,u_j]] \in V_{2m}$, and $A_m \in V_{2m+1}$, and since $B_{mj} = b_{mj}$ and $a_m = A_m$, the conclusion follows immediately.

The following lemma allows us to draw further results from the preceding lemmas.

5.5.11 <u>Lemma</u>: Let $V$ be a fully invariant subgroup of $H$, $J$ a finite subset of $I$, $b_j \in \gamma_3(H)$ $(j \in J)$, $u_j \in \underline{\underline{h}}(j \in J)$ and $v \in \underline{\underline{h}} \setminus E(b_j : j \in J) \cup \{u_j : j \in J\}$. Then if $\prod_{j \in J} [b_j,[v,u_j]] \in V$, it follows that

$$[b_j,[v,u_j]] \in V \quad \text{for each } j \in J.$$

<u>Proof</u>: For each $j \in J$, let $\phi_j$ be the endomorphism of $H$ determined by the action:

$$v\phi_j = u_j$$
$$u\phi_j = u \quad \text{for } u \in \underline{\underline{h}} \setminus v.$$

Then $b_j\phi_j = b_j$ for all $j \in J$, and if $k \in J$,

$$(\prod_{j\in J} [b_j,[v,u_j]])\phi_k = \prod_{\substack{j\in J\\ j\neq k}} [b_j,[v,u_j]],$$

and hence

$$[b_k,[v,u_k]] = \prod_{j\in J} [b_j,[v,u_j]]((\prod_{j\in J}[b_j,[v,u_j]])\phi_k)^{-1} \in V.$$

5.5.12 Lemma: Let $V$ be a fully invariant subgroup of $H$, $J$ a finite subset of $I$, and let

$$w = \prod_{i=1}^{p-1} [a_i,iv] \prod_{j\in J} \{[b_{oj},[v,u_j]][b_{ij},[v,u_j],iv]\},$$

where $a_1,\ldots,a_{p-1} \in [\gamma_3(H),\gamma_2(H)]$, $b_{ij} \in \gamma_3(H)$ $(0 \leq i \leq p-1, j \in J)$, $u_j \in \underline{\underline{h}}$ $(j \in J)$, and $v \in \underline{\underline{h}}\backslash E(a_i,b_{oj},b_{ij} : 1 \leq i \leq p-1, j \in J) \cup \{u_j : j\in J\}$.

Then there exist integers $s_i \in I$ $(0 \leq i \leq p-1)$, and $r_k \in I$ $(1 \leq k \leq p-1)$ such that

$$[b_{ij},[v,u_j]] \in \langle w\rangle_{s_i} \text{ for each } j \in J, \text{ and } a_k \in \langle w\rangle_{r_k}.$$

Proof: In 5.5.10 put $m = p-1$ and $V = \langle w\rangle$. Then it follows immediately that

5.5.13 $$\prod_{j\in J} [b_{p-1,j},[v,u_j]] \in \langle w\rangle_{2(p-1)}$$

and

$$a_{p-1} \in \langle w\rangle_{2p-1}.$$

Then by applying 5.5.11 we have that $[b_{p-1,j},[v,u_j]] \in \langle w\rangle_{2(p-1)}$ for each $j \in J$. From 5.5.13 we also have

$$\prod_{i=1}^{p-2} [a_i,iv] \prod_{j\in J} \{[b_{oj},[v,u_j]][b_{ij},[v,u_j],iv]\} \in \langle w\rangle_p.$$

If we now employ 5.5.10 again, but this time with $m = p - 2$ and $V = \langle w\rangle_p$ we obtain the result that $\prod_{j\in J} [b_{p-2,j},[v,u_j]] \in \langle w\rangle_{s_{p-2}}$ where $s_{p-2} = 3p-4$, and $a_{p-2} \in \langle w\rangle_{r_{p-2}}$ where $r_{p-2} = s_{p-2} + 1$. Applying 5.5.11 again we obtain the assertion of the lemma that $[b_{p-2,j},[v,u_j]] \in \langle w\rangle_{s_{p-2}}$ for all $j \in J$. With another $p - 2$ applications of this procedure we obtain the assertion of the lemma.

5.5.14 <u>Lemma</u>: Let $\underline{D} = \{0,1,\ldots,p-1\}^s$ so that each $\underline{d} \in \underline{D}$ is an s-tuplet $\underline{d} = (d_1,\ldots,d_s)$ with $0 \leq d_i \leq p-1$ for $i = 1,\ldots,s$. Let $\underline{I} = \{((i_1,i_2),\underline{d}) : i_1,i_2 \in I^+,\ i_1 > i_2,\ \underline{d} \in D\}$, and let $\underline{J}$ be a finite subset of $\underline{I}$. Let

$$w = \prod_{\underline{i}\in\underline{J}} [w_{\underline{i}},[a_{i_1},a_{i_2}],d_1a_1,\ldots,d_sa_s]$$

where $w_{\underline{i}} \in \gamma_3(H)$ for all $\underline{i} \in \underline{J}$,

$\{a_1,\ldots,a_s\} \cup \{a_{i_1},a_{i_2} : ((i_1,i_2),\underline{d}) \in \underline{J}\} \subseteq \underline{\underline{h}}\backslash E(w_{\underline{i}} : \underline{i} \in \underline{J})$,

and $E([w_{\underline{i}},[a_{i_1},a_{i_2}],d_1a_1,\ldots,d_sa_s]) = E(w)$ for all $\underline{i} \in \underline{J}$.

Then for each $\underline{i} \in \underline{J}$, there exists an integer $e_{\underline{i}} \in I$ such that

$$[w_{\underline{i}},[a_{i_1},a_{i_2}]] \in \langle w\rangle_{e_{\underline{i}}}.$$

<u>Proof</u>: The proof is by induction on s. For $s = 1$ we may write $w = \prod_{\underline{i}\in\underline{J}} [w_{\underline{i}},[a_{i_1},a_{i_2}],d_1a_1]$. Since $\{a_{i_1},a_{i_2} : ((i_1,i_2),\underline{d}) \in J\}$ $\not\subseteq E(w_{\underline{i}} : \underline{i} \in \underline{J})$ and using the assumption that

$E([w_{\underline{i}},[a_{i_1},a_{i_2}],d_1a_1]) = E(w)$ for each $\underline{i} \in \underline{J}$, we deduce that $w$ can be written in one of the following ways:

(i) $\quad w = \prod_{d_1=0}^{p-1} [w_{d_1},[a_k,a_1],d_1a_1]$ for some $k \in I^+\backslash\{1\}$,

(ii) $w = \prod_{d_1=1}^{p-1} [w_{d_1},[a_k,a_j],d_1a_1]$ for some $k,j \in I^+\setminus\{1\}$,

or (iii) $w = [w_0,[a_k,a_j]]$ for some $k,j \in I^+\setminus\{1\}$.

where $w_{d_1} = \begin{cases} w_{\underline{i}} & \text{if } ((k,j),d_1) \in \underline{J} \\ 1 & \text{if } ((k,j),d_1) \notin \underline{J} \end{cases}$ .

Using 5.5.12 we conclude that for each $\underline{i} \in \underline{J}$ there is an integer $e_{\underline{i}}$ such that $[w_{\underline{i}},[a_{i_1},a_{i_2}]] \in \langle w\rangle_{e_{\underline{i}}}$.

Now assume the result for $s' < s$. For each $d_s \in \{0,1,\ldots,p-1\}$, set

$$\underline{D}_{d_s} = \{(d_1',\ldots,d_s') \in \underline{D} \mid d_s' = d_s\},$$

$$\underline{J}(d_s,0) = \{((i_1,i_2),\underline{d}) \in \underline{J} \mid \underline{d} \in \underline{D}_{d_s} \text{ and } i_1 \neq s \neq i_2\},$$

$$\underline{J}(d_s,j) = \{((i_1,i_2),\underline{d}) \in \underline{J} \mid \underline{d} \in \underline{D}_{d_s} \text{ and } (i_1,i_2) = (s,j) \text{ or } (j,s)\},$$

and let

$$w(d_s,0) = \prod_{\underline{i}\in\underline{J}(d_s,o)} [w_{\underline{i}},[a_{i_1},a_{i_2}],d_1a_1,\ldots,d_{s-1}a_{s-1}],$$

$$w(d_s,j) = \prod_{\underline{i}\in\underline{J}(d_s,j)} [w_{\underline{i}},d_1a_1,\ldots,d_{s-1}a_{s-1}]^{\varepsilon_{\underline{i}}}$$

where $\varepsilon_{\underline{i}} = \begin{cases} 1 & \text{if } ((i_1,i_2),\underline{d}) = ((s,j),\underline{d}) \\ -1 & \text{if } ((i_1,i_2),\underline{d}) = ((j,s),\underline{d}). \end{cases}$

Let $K = \{j \in I \mid \underline{J}(d_s,j) \neq \emptyset\}$. Then we can write $w$ in the form

$$w = \prod_{d_s=1}^{p-1} [w(d_s,0),d_sa_s] \prod_{j\in K} \{[w(d_s,j),[a_s,a_j]][w(d_s,j),[a_s,a_j],d_sa_s]\}.$$

Then by 5.5.12 there exist integers $e(d_s,0)$ for $d_s \in \{1,\ldots,p-1\}$ and $e(d_s,j)$ for $d_s \in \{0,1,\ldots,p-1\}$, $j \in K$ such that

$$w(d_s,0) \in \langle w\rangle_{e(d_s,0)} ,$$

and $[w(d_s,j),[a_s,a_j]] \in \langle w\rangle_{e(d_s,j)}$ .

By the inductive hypothesis, for each $\underline{i} \in \underline{J}(d_s,0)$ there exists an integer $r_{\underline{i}}$ such that $[w_{\underline{i}},[a_{i_1},a_{i_2}]] \in \langle w(d_s,0)\rangle_{r_{\underline{i}}}$, and combining this result with the result above we have

$$[w_{\underline{i}},[a_{i_1},a_{i_2}]] \in \langle w(d_s,0)\rangle_{r_{\underline{i}}} \le (\langle w\rangle_{e(d_s,0)})_{r_{\underline{i}}} = \langle w\rangle_{e_{\underline{i}}} ,$$

where

$$e_{\underline{i}} = e(d_s,0) + r_{\underline{i}}.$$

For $\underline{i} \in \underline{J}(d_s,j)$ we have from the above result that

$$\prod_{\underline{i}\in\underline{J}(d_s,j)} [w_{\underline{i}},[a_{i_1},a_{i_2}],d_1a_1,\ldots,d_{s-1}a_{s-1}] \in \langle w\rangle_{e(d_s,j)}$$

for some $j \in K$, and therefore by the inductive hypothesis, there exists an integer $r_{\underline{i}} \in I$ such that

$$[w_{\underline{i}},[a_{i_1},a_{i_2}]] \in (\langle w\rangle_{e(d_s,j)})_{r_{\underline{i}}} = \langle w\rangle_{e_{\underline{i}}}$$

where $e_{\underline{i}} = e(d_s,j) + r_{\underline{i}}$.

5.5.15 <u>Lemma</u>: Let $w$ be a non-trivial homogeneous special element of $\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p))$, for $p \neq 2$, or of $\underline{\underline{A}}_2\underline{\underline{A}}(H(2))$. Then there exists an integer $e$ such that

$$\langle w\rangle \ge [\gamma_3(H(p)),\gamma_2(H(p)),e\,H(p)].$$

<u>Proof</u>: Let $E(w) = \{h_1,h_2,h_3,a_1,\ldots,a_s\}$ and let $w$ be expressed in normal form by

$$w = \prod_{i=1}^{t} [[h_2,h_1,h_3,d_{i1}a_1,\ldots,d_{is}a_s],[a_{i_j},a_{i_k}]]^{e_i}$$

where, $i_j, i_k \in \{1,\ldots,s\}$. Then

$$w = \prod_{i=1}^{t} [[h_2,h_1,h_3],[a_{i_j},a_{i_k}],d_{i1}a_1,\ldots,d_{is}a_s]^{e_i} \quad \text{by 1.2.3(ii),}$$

$$= \prod_{i=1}^{t} [[h_2,h_1,h_3]^{e_i},[a_{i_j},a_{i_k}],d_{i1}a_1,\ldots,d_{is}a_s],$$

and thus in the notation of 5.5.14

$$w = \prod_{\underline{i}\in \underline{J}} [w_{\underline{i}},[a_{i_1},a_{i_2}],d_1a_1,\ldots,d_sa_s],$$

where $\underline{J} = \{((i_j,i_k),(d_{i1},\ldots,d_{is})) : i = 1,\ldots,t\}$ and for $\underline{d} = (d_1,\ldots,d_s)$, $\underline{i} = ((i_1,i_2),\underline{d})$, $w_{\underline{i}}$ is defined, for $\underline{i} \in \underline{J}$, by

$$w_{\underline{i}} = [h_2,h_1,h_3]^{e_i} \text{ if } i_j = i_1,\ i_k = i_2, \text{ and } d_{ij} = d_j \text{ for}$$
$j = 1,\ldots,s$.

Thus by 5.5.14 for each $\underline{i} \in \underline{J}$ there exists an integer $e_{\underline{i}}$ such that $[w_{\underline{i}},[a_{i_1},a_{i_2}]] \in \langle w\rangle_{e_{\underline{i}}}$. Therefore, it follows that $[[h_2,h_1,h_3]^{e_i},[a_{i_j},a_{i_k}]] \in \langle w\rangle_{t_i}$ for some integer $t_i$, and consequently that $[[h_2,h_1,h_3],[a_{i_j},a_{i_k}]] \in \langle w\rangle_{t_i}$. Put $e = \min t_i$. Then, for all $u_1,u_2,u_3,v_1,v_2,w_1,\ldots,w_e \in H$,

$$[[u_1,u_2,u_3],[v_1,v_2],w_1,\ldots,w_e] \in \langle w\rangle,$$

and this implies that $[\gamma_3(H),\gamma_2(H),e\,H] \in \langle w\rangle$.

The next lemma is very short, and after it we will be ready to prove 5.3.5 and 5.3.9.

5.5.16 <u>Lemma</u>: Let $G$ be any group and $V$ a normal subgroup of $G$ such that $[\gamma_3(G),\gamma_2(G),k\,G] \leq V$ for some $k \in I$. Then $[\gamma_2(G),\gamma_2(G),(k+1)G] \leq V$.

<u>Proof</u>: The proof is by induction on $k$. For $k = 0$ we use 1.2.2(iii) which gives

$$[\gamma_2(G),\gamma_2(G),G] \leq [[\gamma_2(G),G],\gamma_2(G)][[\gamma_2(G),G],\gamma_2(G)]$$

$$= [\gamma_3(G),\gamma_2(G)] \leq V.$$

The rest of the induction is routine and is omitted.

<u>Proof of 5.3.5</u>: By 5.5.2 it is sufficient to consider the case when $w$ is a non-trivial homogeneous special element of $\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p))$ for $p \neq 2$. Then by 5.5.15 there is an $e \in I$ such that $\langle w\rangle \geq [\gamma_3(H(p)),\gamma_2(H(p)),e\, H(p)]$, and hence by 5.5.16 $\langle w\rangle \geq [\gamma_2(H(p)),\gamma_2(H(p)),(e+1)H(p)]$. But now 5.2.8 implies that $\langle w\rangle \geq [\underline{\underline{A}}_p\underline{\underline{A}}_p(H(p)),k\, H(p)]$ where $k = e + 2$, which gives the required result.

<u>Proof of 5.3.9</u>: Again we consider the case when $w$ is a non-trivial homogeneous special element of $\underline{\underline{A}}_2\underline{\underline{A}}(H(2))$. By 5.5.15 there is an integer $e \in I$ such that $\langle w\rangle \geq [\gamma_3(H(2)),\gamma_2(H(2)), e\, H(2)]$, and hence by 5.5.16 $\langle w\rangle \geq [\gamma_2(H(2)),\gamma_2(H(2)),(e+1)H(2)]$. But now 5.2.9 implies that $\langle w\rangle \geq [\underline{\underline{A}}_2\underline{\underline{A}}(H(2)),k\, H(2)]$ where $k = e + 2$, and this gives the required result.

We now work through a similar set of lemmas for $K$ before we finally prove 5.3.8.

5.5.17 <u>Lemma</u>: Let $w = [a_o,v^2][a_1,v^2,v][b,v]$ where $a_o,a_1 \in \gamma_3(K)$, $b \in \underline{\underline{A}}_2\underline{\underline{A}}_2(K)$ and $v \in \underline{\underline{k}}\backslash E(b,a_o,a_1)$. Then there exist integers $r_0,r_1$, and $s$ such that $[a_o,v^2] \in \langle w\rangle_{r_0}$, $[a_1,v^2] \in \langle w\rangle_{r_1}$ and $b \in \langle w\rangle_s$.

Proof: For any $k \in K$ there is an endomorphism $\theta$ of $K$ such that $a_i\theta = a_i (i = 0,1)$, $b\theta = b$ and $v\theta = k$, so it follows that

5.5.18 $$[a_o, k^2][a_1, k^2, k][b,k] \in \langle w \rangle$$

for all $k \in K$. Therefore, for $x \in \underline{\underline{k}} \backslash E(a_o, a_1, b)$, $x \neq v$,

$$[a_o, (vx)^2][a_1, (vx)^2, vx][b, vx] \in \langle w \rangle.$$

Expanding this and applying 5.5.18 we have

5.5.19 $$[a_1, v^2, x][a_1, v^2, v, x][a_1, x^2, v] \times [a_1, x^2, x, v][b, x, v] \in \langle w \rangle.$$

In 5.5.19 replace v by vy where $y \in \underline{\underline{k}} \backslash E(a_o, a_1, b)$, $x \neq y \neq v$, and expand as before. Applying 5.5.19 we get

$$[a_1, v^2, y, x][a_1, v^2, v, y, x][a_1, y^2, v, x][a_1, y^2, v, y, x] \times [a_1, x^2, v, y][a_1, x^2, x, v, y][b, x, v, y] \in \langle w \rangle.$$

But by 5.5.19

$$[a_1, x^2, v, y][a_1, x^2, x, v, y][a_1, y^2, v, x] \times [a_1, y^2, v, y, x][b, x, v, y] \in \langle w \rangle,$$

so we conclude that

$$[a_1, v^2, y, x][a_1, v^2, v, y, x] \in \langle w \rangle$$

or

$$[a_1, v^2, y, x]^v \in \langle w \rangle$$

and hence

$$[a_1, v^2, y, x] \in \langle w \rangle$$

for $x, y \in \underline{\underline{k}} \backslash E(a_1)$. Thus $[a_1, v^2] \in \langle w \rangle_{r_1}$ where $r_1 = 2$.

We can also conclude that $[a_0,v^2][b,v] \in \langle w\rangle_1$. If we replace $v$ by $vx$ for $x \in \underline{\underline{k}}\backslash E(a_0,b)$, $x \neq v$, and expand the result we get $[b,v,x] \in \langle w\rangle_1$, and hence $b \in \langle w\rangle_s$ where $s = 3$. From this we deduce that $[a_0,v^2] \in \langle w\rangle_{r_0}$ where $r_0 = s$.

5.5.20 <u>Lemma</u>: Let $w \in K$ such that

$$w = [u,a_1,\ldots,a_s] \prod_{i=1}^{s} [w_i,a_i^2,a_1,\ldots,a_s][v_i,a_i^2,a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_s]$$

where $u \in \underline{\underline{A}}_2\underline{\underline{A}}_2(K)$, $w_i,v_i \in \gamma_3(K)$ $(1 \leq i \leq s)$, and $\{a_1,\ldots,a_s\} \in \underline{\underline{k}}\backslash E(u,w_i,v_i : i = 1,\ldots,s)$. Then there exist integers $n,r_i,t_i$ for $i \in \{1,\ldots,s\}$ such that

$$u \in \langle w\rangle_n,\quad [w_i,a_i^2] \in \langle w\rangle_{r_i} \quad\text{and}\quad [v_i,a_i^2] \in \langle w\rangle_{t_i}.$$

<u>Proof</u>: The proof is by induction on $s$. For $s = 1$ the lemma reduces to 5.5.17. For $s > 1$, put

$$W_s = [w_s,a_1,\ldots,a_{s-1}],$$

$$V_s = [v_s,a_1,\ldots,a_{s-1}] \quad\text{and}$$

$$W = [u,a_1,\ldots,a_s] \prod_{i=1}^{s-1} [w_i,a_i^2,a_1,\ldots,a_{s-1}] \times [v_i,a_i^2,a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_{s-1}].$$

Then we can write $w$ as follows:

$$w = [W,a_s][W_s,a_s^2,a_s][V_s,a_s^2].$$

We can now apply 5.5.17 to find integers $e_s$, $f_s$ and $e$ such that

$$[W_s,a_s^2] \in \langle w\rangle_{e_s},\quad [V_s,a_s^2] \in \langle w\rangle_{f_s}$$

and

$$W \in \langle w\rangle_e.$$

By the inductive hypothesis, for each $i \in \{1,\ldots,s-1\}$ there exist integers $e_i$, $f_i$ and $j$ such that

$$[w_i,a_i^2] \in \langle W\rangle_{e_i},\ [v_i,a_i^2] \in \langle W\rangle_{f_i} \quad \text{and}$$

$$u \in \langle W\rangle_j .$$

Combining these results with those above we have

$$u \in \langle w\rangle_n,\quad [w_i,a_i^2] \in \langle w\rangle_{r_i},\quad \text{and}\quad [v_i,a_i^2] \in \langle w\rangle_{t_i}$$

where $n = j + e$, $r_i = e_i + e$ and $t_i = k_i + e$ $(1 \le i \le s\text{-}1)$.

We also have that $[W_s,a_s^2] \in \langle w\rangle_{e_s}$ which we can write as

$$[w_s,a_s^2,a_1,\ldots,a_{s-1}] \in \langle w\rangle_{e_s},$$

and then by the inductive hypothesis there is an integer $m$ such that

$$[w_s,a_s^2] \in (\langle w\rangle_{e_s})_m = \langle w\rangle_{r_s}$$

where $r_s = e_s + m$. In the same way there is an integer $m'$ such that

$$[v_s,a_s^2] \in (\langle w\rangle_{f_s})_{m'} = \langle w\rangle_{t_s}$$

where $t_s = k_s + m'$.

<u>Proof of 5.3.8</u>: By 5.5.2 it is sufficient to consider the case when $w$ is a non-trivial homogeneous special element of $\underline{\underline{A}}_2\underline{\underline{A}}_2(K)$. Let $E(w) = \{k_1,k_2,k_3,a_1,\ldots,a_s\}$, and let $w$ be expressed in normal form by

$$w = \prod_{i=1}^{s} [[k_2,k_1,k_3,a_1,\ldots,a_s],a_i^2]^{e_i}$$

$$\times [[k_2,k_1,k_3,a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_s],a_i^2]^{d_i}$$

where $e_i,d_i \in \{0,1,\ldots,p\text{-}1\}$ for $i = 1,\ldots,s$. Then

$$w = \prod_{i=1}^{s} [[k_2,k_1,k_3],a_i^2,a_1,\ldots,a_s]^{e_i}$$

$$\times [[k_2,k_1,k_3],a_i^2,a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_s]^{d_i}$$

$$= \prod_{i=1}^{s} [w_i, a_i^2, a_1, \ldots, a_s][v_i, a_i^2, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_s]$$

where $w_i = [k_2, k_1, k_3]^{e_i}$ and $v_i = [k_2, k_1, k_3]^{d_i}$. Thus we can apply 5.5.20 to find integers $r_i$ and $t_i$ such that

$$[w_i, a_i^2] \in \langle w \rangle_{r_i} \quad \text{and} \quad [v_i, a_i^2] \in \langle w \rangle_{t_i}.$$

Therefore for $e_i \neq 0$, $d_i \neq 0$ it follows that

$$[[k_2, k_1, k_3]^{e_i}, a_i^2] \in \langle w \rangle_{r_i} \quad \text{and}$$

$$[[k_2, k_1, k_3]^{d_i}, a_i^2] \in \langle w \rangle_{t_i}.$$

Put $t = \min\{r_i, t_i\}$. Then for all $u, v, w, x, y_1, \ldots, y_t \in K$ we have

$$[[u, v, w], x^2, y_1, \ldots, y_t] \in \langle w \rangle,$$

and therefore

$$[[u, v], x^2, w, y_1, \ldots, y_t] \in \langle w \rangle.$$

But this implies that $[\underline{\underline{A}}(K), \underline{\underline{A}}_2(K), (t+1)K] \leq \langle w \rangle$, and therefore by 5.2.10

$$[\underline{\underline{A}}_2\underline{\underline{A}}_2(K), e\,K] \leq \langle w \rangle,$$

where $e = t + 2$.

## 5.6 Some Further Remarks.

These results give some description of the subvarieties of $\underline{\underline{A}}_p\underline{\underline{T}}_p \wedge \underline{\underline{T}}_p\underline{\underline{A}}_p$, but the question remains of what can be said in general of the subvarieties of $\underline{\underline{A}}_p\underline{\underline{T}}_p \wedge \underline{\underline{N}}_c\underline{\underline{A}}_p$, for $c > 2$, and of $\underline{\underline{A}}_p\underline{\underline{T}}_p$. We do know that every subvariety of $\underline{\underline{A}}_p\underline{\underline{T}}_p \wedge \underline{\underline{N}}_c\underline{\underline{A}}_p$ has a finite basis for its laws, but even this is not known for the subvarieties of $\underline{\underline{A}}_p\underline{\underline{T}}_p$.

The calculations used in Chapter 5 were quite complicated and it appears that to proceed further, using similar methods, the calculations would be so complicated that their value would be doubtful. It is also not clear what results could be expected. So it seems that further results concerning the subvarieties of $\underline{\underline{A}}_p\underline{\underline{T}}_p$ would be quite difficult to find and possibly need a different approach.

## REFERENCES

[1] E. Artin, Geometric Algebra (Interscience Publishers Inc., New York, 1957).

[2] J. M. Brady, On the classification of just-non-Cross varieties of groups, Bull. Austral. Math. Soc. 3 (1970), 293-311.

[3] J. M. Brady, R. A. Bryce and John Cossey, On certain abelian - by - nilpotent varieties, Bull. Austral. Math. Soc. 1 (1969), 403-416.

[4] Warren Brisley, On varieties of metabelian p-groups and their laws, J. Austral. Math. Soc. 7 (1967), 64-80.

[5] Warren Brisley, Varieties of metabelian p-groups of class p, p+1, J. Austral. Math. Soc. 12 (1971), 53-62.

[6] M. S. Brooks, On varieties of metabelian groups of prime-power exponent. Ph.D. Thesis, A.N.U. (1968).

[7] M. S. Brooks, On lattices of varieties of metabelian groups, J. Austral. Math. Soc. 12 (1971), 161-166.

[8] M. S. Brooks, On varieties of metabelian groups of prime-power exponent, J. Austral. Math. Soc. 14 (1972), 129-154.

[9] R. M. Bryant and M. F. Newman, Some finitely based varieties of groups, Proc. London Math. Soc., to appear.

[10] R. A. Bryce, Metabelian groups and varieties, Philos. Trans. Roy Soc. London, Ser. A. 266 (1970), 281-355.

[11] D. E. Cohen, On the laws of a metabelian variety, J. Algebra 5 (1967), 267-273.

[12] Marshall Hall, Jr., The Theory of Groups (The Macmillan Company, New York, 1959).

[13] Graham Higman, Representations of general linear groups and varieties of groups, Proc. Internat. Conf. Theory of Groups. Austral. Nat. Univ., Canberra 1965 (Gordon and Breach, New York, 1967).

[14] L. G. Kovács and M. F. Newman, On non-Cross varieties of groups, J. Austral. Math. Soc. 12 (1971), 129-144.

[15] Hans Liebeck, Concerning nilpotent wreath products, Proc. Cambridge Philos. Soc. 58 (1962), 443-451.

[16] Wilhelm Magnus, Abraham Karass and Donald Solitar, Combinatorial Group Theory (Interscience Publ. Inc., New York, 1966).

[17] Hanna Neumann, Varieties of Groups (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[18] M. R. Vaughan-Lee, Abelian by nilpotent varieties, Quart. J. Math. Oxford (2) 21 (1970), 193-202.

[19] Paul M. Weichsel, On metabelian p-groups, J. Austral. Math. Soc. 7 (1967), 55-63.

# APPENDIX

Table 1 is a list of commutator elements that form a basis for $\gamma_6(G)$ where $G = F_3(\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_6)$. The usual convention is followed of representing $g_i$ by $i$. Tables 2 and 3 show the action of the generating automorphisms $\beta$, $\rho_2$ and $\rho_3$ on the subgroups defined in section 3.2 of chapter 3.

Table 1

| | | |
|---|---|---|
| $c_1 = [2,1,1,2,2,3]$ | $c_9 = [3,1,1,1,1,2]$ | $c_{18} = [2,1,1,1,1,2]$ |
| $c_2 = [2,1,1,2,3,3]$ | $c_{10} = [3,1,1,2,3,3]$ | $c_{19} = [2,1,1,1,2,2]$ |
| $c_3 = [2,1,1,3,3,3]$ | $c_{11} = [3,1,1,2,2,2]$ | $c_{20} = [2,1,1,2,2,2]$ |
| $c_4 = [2,1,1,1,2,3]$ | $c_{12} = [3,1,1,2,2,3]$ | $c_{21} = [3,1,1,1,1,3]$ |
| $c_5 = [2,1,1,1,3,3]$ | $c_{13} = [3,1,1,1,2,3]$ | $c_{22} = [3,1,1,1,3,3]$ |
| $c_6 = [2,1,2,2,2,3]$ | $c_{14} = [3,1,1,1,2,2]$ | $c_{23} = [3,1,1,3,3,3]$ |
| $c_7 = [2,1,2,2,3,3]$ | $c_{15} = [3,1,2,2,2,3]$ | $c_{24} = [3,2,2,2,2,3]$ |
| $c_8 = [2,1,2,3,3,3]$ | $c_{16} = [3,1,2,2,3,3]$ | $c_{25} = [3,2,2,2,3,3]$ |
| | $c_{17} = [3,1,2,3,3,3]$ | $c_{26} = [3,2,2,3,3,3]$. |

Table 2

| $g$ | $g\beta$ | $g\rho_2$ | $g\rho_3$ |
|---|---|---|---|
| $u_1$ | $u_1u_2$ | $u_2$ | $u_3$ |
| $u_2$ | $u_2$ | $u_1$ | $w_2u_2$ |
| $u_3$ | $w_7u_3$ | $w_3u_3$ | $u_1$ |
| $v_1$ | $av_1v_2$ | $v_2$ | $v_3$ |
| $v_2$ | $v_2$ | $v_1$ | $w_2v_2$ |
| $v_3$ | $w_{17}v_3$ | $w_3v_3$ | $v_1$ |

Note : $a = w_1w_2w_6w_{10}$ .

The deletions act trivially on $u_1,u_2,u_3,v_1,v_2$ and $v_3$.

Table 3

| $g$ | $g\beta$ | $g\rho_2$ | $g\rho_3$ |
|---|---|---|---|
| $w_1$ | $w_1w_2$ | $w_2$ | $w_3$ |
| $w_2$ | $w_2$ | $w_1$ | $w_2$ |
| $w_3$ | $w_3$ | $w_3$ | $w_1$ |
| $w_4w_7$ | $w_4w_7$ | $w_5w_8$ | $w_6w_9$ |
| $w_5w_8$ | $w_3w_4w_7w_5w_8$ | $w_4w_7$ | $w_5w_8$ |
| $w_6w_9$ | $w_2w_6w_9$ | $w_6w_9$ | $w_4w_7$ |

# CORRIGENDA

The proof of 2.1.4 (p14) is not adequate. It is proved as a special case of 35.21 of [17]. This is unnecessary and a suitable proof can be given as follows.

2.1.4 Lemma : If $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_{c+1}$ is generated by its free group of rank $r$, then $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$ is also generated by its free group of rank $r$.

Proof : Suppose $G_r(\alpha,c)$ does not generate $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$. Then there is a word $w$ that is a law in $G_r(\alpha,c)$ but not in $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c$, and we may assume that $w$ is a word in $s$ variables, where $s \leq c$. Form $w^* = [w(x_1,\ldots,x_s),x_{s+1}]$. Then $w^*$ is a law in $G_r(\alpha,c+1)$.

Let $w\theta$ be a non-trivial value of $w$ in $G_c(\alpha,c)$. Let $\theta^* \in \mathrm{Hom}(X,G_{c+1}(\alpha,c+1))$ such that $x_i\theta^* = x_i\theta$ for $i \neq s+1$, and $x_{s+1}\theta^* = g_{c+1}$. Then $w^*\theta^* = [w\theta,g_{c+1}] \neq 1$. Thus $w^*$ is a law in $G_r(\alpha,c+1)$, but is not a law in $G_{c+1}(\alpha,c+1)$ and therefore not a law in $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_{c+1}$ which contradicts our original assumption.

The statement and proof of 2.2.3 (p22) is also unsatisfactory, and we shall prove a slightly different version of this lemma. First there are some preliminary results.

Lemma A : Let $G = F_r(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha})$. Then $G$ is nilpotent of class $r(p^\alpha - 1) + 1$.

Proof : By 22.48 of [17] $G$ can be embedded in $F_r(\underline{\underline{A}}_p) \text{ wr } F_r(\underline{\underline{A}}_{p^\alpha})$, and by 5.1 of [15] this wreath product has class $r(p^\alpha - 1) + 1$, so that the class of $G$ is not greater than $r(p^\alpha - 1) + 1$.

By 1.1.7 $G$ has non-trivial commutator elements of weight $r(p^\alpha - 1) + 1$, and we conclude that the class of $G$ is exactly $r(p^\alpha - 1) + 1$.

Corollary : $\gamma_c(G_r(\alpha,c)) \neq \{1\}$ if $c \leq r(p^\alpha - 1) + 1$.

Before stating the next lemma we recall that in 2.1.2 we defined the weight of an element $u \in G'_r(\alpha,c)$. However, we have not assigned a weight to the identity element, and we do this now by saying $wt(1) = \omega$.

Lemma B : Let $r,c \in I^+$ such that $\gamma_c(G_r(\alpha,c)) \neq \{1\}$, and let $x$ be an element of $G'_r(\alpha,c)$ such that $wt\, x = k < c$. Then there is an $i \in \{1,\dots,r\}$ such that $[x,g_i] \neq 1$, and $wt[x,g_i] = k+1$.

Proof : Let $x$ be expressed in normal form by $x = b_1^{e_1}\dots b_t^{e_t}$ where $b_j = [u_j,v_j,\delta_j]$, $j = 1,\dots,t$, and we may assume that $k = wt\, b_1 \leq wt\, b_2 \leq \dots \leq wt\, b_t$.

We proceed to choose $i$ as follows.

If $\delta_1(g_j) > 0$ for all $j \in \{1,\dots,r\}$, then either there is an $m \in \{1,\dots,r\}$ such that $\delta_1(g_m) < p^\alpha - 1$, in which case we put $i = m$, or $\delta_1(g_j) = p^\alpha - 1$, for all $j \in \{1,\dots,r\}$, since $\gamma_c(G_r(\alpha,c)) \neq \{1\}$ and $k < c$. In this latter case we choose $i$ such that $g_i = u_1$. In either case, $b'_1 = [b_1,g_i]$ is again a basis element and $wt\, b'_1 = k+1$.

If $\delta_1(g_j) = 0$ for some $j \in \{1,\dots,r\}$, we put $i = j$. Then $b'_1 = [b_1,g_i] \neq 1$, and if $g_i > v_1$, $b'_1$ is again a basis element and $wt\, b'_1 = k+1$. If $g_i < v_1$ then

$$
\begin{aligned}
b'_1 &= [[u_1,v_1,\delta_1], g_i] \\
&= [u_1,g_i,\delta'_1][v_1,g_i,\delta'_1]^{-1}
\end{aligned}
$$

where for $j \neq i$, $\delta'_1(g_j) = \delta_1(g_j)$ and $\delta'_1(g_i) = 1$. In this case $b'_1$ is written as a product of basis elements and $wt\, b'_1 = k+1$.

We now have to show that $x' = [x,g_i] \neq 1$. But

$$
\begin{aligned}
x' &= [b_1^{e_1}\dots b_t^{e_t}, g_i] \\
&= \prod_{j=1}^{t} [b_j^{e_j}, g_i] \\
&= \prod_{j=1}^{t} [b_j, g_i]^{e_j} \\
&= b_1'^{\,e_1}\, x''.
\end{aligned}
$$

Then $x''$ can be written as a product of basis elements

$x'' = d_1^{f_1} \cdots d_s^{f_s}$ where $\operatorname{wt} d_j \geqslant k+1$, for $j = 1,\ldots,s$. If $b_1'$ is a basis element, then by our choice of $g_i$, $b_1' \neq d_j$ for any $j \in \{1,\ldots,s\}$ and we conclude that $x' \neq 1$.

Otherwise $b_1' = [u_1, g_i, \delta_1'] [v_1, g_i, \delta_1']^{-1}$ and again by our choice of $g_i$, $[u_1, g_i, \delta_1'] \neq d_j$ for any $j \in \{1,\ldots,s\}$, and we conclude that $[x, g_i] \neq 1$.

Thus in both cases we have that $[x, g_i] \neq 1$, and since $\operatorname{wt} b_1' = k+1$ and $\operatorname{wt} x'' \geqslant k+1$, we conclude that $\operatorname{wt} [x, g_i] = k+1$.

We can now restate 2.2.3 and give a suitable proof.

2.2.3 Lemma : Let $c, r, r' \in I^+$ such that $r' < r$ and $\gamma_c(G_{r'}(\alpha,c)) \neq \{1\}$ and let $w$ be a law in $G_{r'}(\alpha,c)$ but not in $G_r(\alpha,c)$. Then there is a word $w_1$ that is a law in $G_{r'}(\alpha,c)$ but not in $G_r(\alpha,c)$ such that all the values of $w_1$ in $G_r(\alpha,c)$ lie in $\gamma_c(G_r(\alpha,c))$.

Proof : Assume $w$ is a word in $n$ variables. If all the values of $w$ in $G_r(\alpha,c)$ lie in $\gamma_c(G_r(\alpha,c))$, then we put $w = w_1$ and we are finished.

Otherwise, let $k = \min\{\operatorname{wt} w\theta : \theta \in \operatorname{Hom}(X, G_r(\alpha,c))\}$ and let $w\phi$ be a value of $w$ in $G_r(\alpha,c)$ such that $\operatorname{wt} w\phi = k$. By lemma B there is an $i \in \{1,\ldots,r\}$ such that $[w\phi, g_i] \neq 1$ and $\operatorname{wt} [w\phi, g_i] = k+1$. But

$$[w\phi, g_i] = [w, x_{n+1}]\phi'$$

where $\quad x_j\phi' = x_j\phi \quad$ for $j \neq n+1$

and $\quad x_{n+1}\phi' = g_i$.

Then $w' = [w, x_{n+1}]$ is also a law in $G_{r'}(\alpha,c)$ but not in $G_r(\alpha,c)$ and $\min\{\operatorname{wt} w'\theta : \theta \in \operatorname{Hom}(X, G_r(\alpha,c))\} = k+1$.

If $k+1 = c$ we put $w' = w_1$. If $k+1 < c$ we can repeat this procedure until we have found $w_1$ so that

$$\min \{ \text{wt } w_1\theta : \theta \in \text{Hom}(X, G_r(\alpha, c))\} = c,$$

and this concludes the proof.

An immediate corollary to Lemma B above is the following:

Corollary C: Let $r, c \in I^+$ such that $\gamma_c(G_r(\alpha,c)) \neq 1$. Then $G_r'(\alpha,c) \cap Z(G_r(\alpha,c)) = \gamma_c(G_r(\alpha,c))$

This corollary and the corrected proof of 2.2.3 also strengthen the proof of 2.4.7. It should be noted that this corrected proof of 2.2.3 shows that the non-trivial values of the word $w_1$ in $G_r(\alpha,c)$ actually have weight $c$, and so in the proof of 2.4.7 we assume that the non-trivial values of $w$ in $G_{r+1}(\alpha,c)$ are in $\gamma_c(G_{r+1}(\alpha,c))$ and that they have weight $c$. The rest of the proof of 2.4.7 follows as in the thesis.